Separation between quantum Lovász number and entanglement-assisted zero-error classical capacity

Xin Wang

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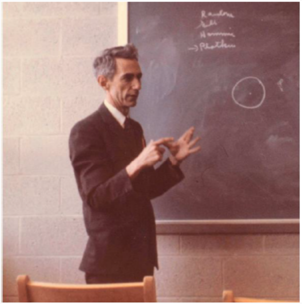
Joint work with **Runyao Duan** (UTS), *arXiv: 1608.04508*

AQIS 2016, Taipei

Introduction

Capacity of a communication channel



Claude Elwood Shannon

(1916 - 2001)

(photo on 17.04.1961)

claude shannon

1916-2016

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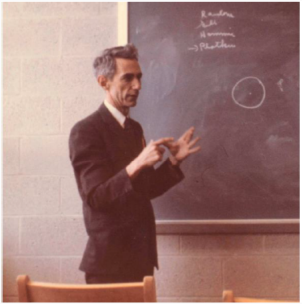

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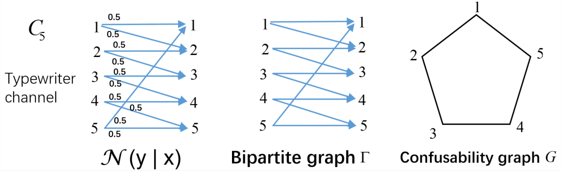
The fundamental problem of communication is that of reproducing at one point, either exactly or approximately, a message selected at another point.
—Claude Elwood Shannon, 1948

- Approximately (small-error) [Shannon-1948]: $\forall m, \widehat{m} \approx m$
- Exactly (zero-error) [Shannon-1956]: $\forall m, \widehat{m} = m$

Zero-error capacity (Shannon capacity) of a classical channel

- Classical channel: $\mathcal{N}(y|x)$, e.g.,



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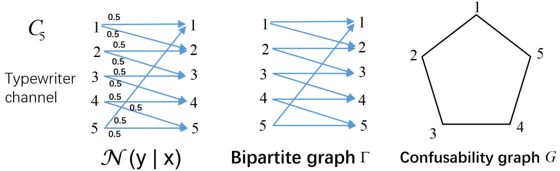


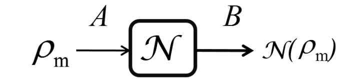
- One-shot zero-error capacity $\alpha(\mathcal{N})$: the maximum number of messages can be sent via $\mathcal{N}$ without error, depends on G.
- Asymptotic zero-error capacity: $C_0(\mathcal{N}) = \sup_{n\geq 1} \frac{1}{n} \log \alpha \left(\mathcal{N}^{\otimes n}\right)$.
- Example:

$$\alpha(C_5) = 2, \alpha(C_5^{\otimes 2}) = 5 \text{ [Shannon-1956]}, C_0(C_5) = \log \sqrt{5} \text{ [Lovász-1979]}.$$

- If there is feedback assistance, the zero-error capacity depends only on the bipartite graph [Shannon-1956].

Zero-error capacity of a quantum channel

- One-shot zero-error classical capacity: The maximum integer n such that $\{\mathcal{N}(\rho_1), \ldots, \mathcal{N}(\rho_n)\}$ are orthogonal (perfectly distinguished).



- **Quantum Channel**: completely positive (CP) and trace-preserving (TP) linear map $\mathcal{N}$ from A to B.
  - Choi-Kraus Representation: $\mathcal{N}(\rho) = \sum_k E_k \rho E_k^\dagger$, $\sum_k E_k^\dagger E_k = \mathbb{1}$
  - Choi-Jamiołkowski matrix: $J_{\mathcal{N}} = \sum_{ij} |i\rangle\langle j|_A \otimes \mathcal{N}(|i\rangle\langle j|_{A'}) = (\mathrm{id}_A \otimes \mathcal{N})|\Phi_{AA'}\rangle\langle\Phi_{AA'}|$, with $|\Phi_{AA'}\rangle = \sum_k |k_A\rangle|k_{A'}\rangle$

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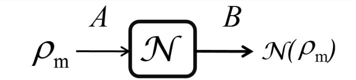

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    $|\Phi_{AA'}\rangle = \sum_k |k_A\rangle|k_{A'}\rangle$
  - Non-commutative (confusability) graph [Duan-Severini-Winter-13]:

    $$S = \mathrm{span}\{E_k^\dagger E_j\} \; (S = S^\dagger, \mathbb{1} \in S).$$

    The zero-error capacity of $\mathcal{N}$ depends only on S.
  - Non-commutative bipartite graph [Duan-Severini-Winter-16]:

    $$K = K(\mathcal{N}) = \mathrm{span}\{E_k\}.$$

    If there is feedback assistance, the zero-error capacity depends only on K.

Quantum Lovász number

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- Nevertheless, [Lovász-1979] introduce the Lovász number to upper bound the zero-error capacity of a classical channel:

$$2^{C_0(\mathcal{N})} \leq \vartheta(G) = \max\{\|\mathbb{1}+T\| : T_{i,j} = 0 \text{ if } \{i,j\} \in E, T = T^\dagger, \mathbb{1}+T \geq 0\}.$$

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- [Duan-Severini-Winter-2013] introduced the quantum Lovász number to upper bound the zero-error capacity of a quantum channel:

$$2^{C_0(\mathcal{N})} \leq \widetilde{\vartheta}(S) = \max \langle\Phi|(\mathbb{1} \otimes \rho + T)|\Phi\rangle$$
$$\text{s.t.} \quad T \in S^\perp \otimes \mathcal{L}(A'), \operatorname{Tr}\rho = 1, \mathbb{1} \otimes \rho + T \geq 0, \rho \geq 0.$$

Quantum zero-error information theory

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- More capacities in quantum zero-error information theory
  - Quantum assistance
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    - Quantum No-signalling Correlations (QNSC)
    - Feedback (F)
  - Zero-error quantum or private capacity [Leung-Yu-2015, next talk!]

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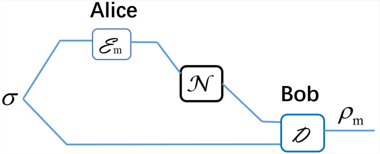
    - Zero-error quantum or private capacity [Leung-Yu-2015, next talk!]
- Entanglement-assisted zero-error classical capacity
    - Make use of (pre-shared) entanglement for communication



    - One-shot entanglement-assisted zero-error capacity $\widetilde{\alpha}(\mathcal{N})$: the maximum number n such that $\{\rho_1, ..., \rho_n\}$ are orthogonal.
    - Asymptotics: $C_{0E}(\mathcal{N}) = \sup_{n\to\infty} \frac{1}{n} \log \widetilde{\alpha}(\mathcal{N}^{\otimes n})$
    - C_{0E} depends on the non-commutative graph S [Duan-Severini-Winter-13].

Separation between quantum Lovász number and entanglement-assisted zero-error classical capacity

Problem: $C_{0E}(\mathcal{N}) = \log \widetilde{\vartheta}(\mathcal{N})$?

Two remarkable phenomena in zero-error information theory

- Entanglement can improve the zero-error capacity
    - Classical channel (no such advantage for the the normal capacity)
        - One-shot: [PRL, Cubitt-Leung-Matthews-Winter-2010]
        - Asymptotic: [CMP, Leung-Mancinska-Matthews-Ozols-Roy-2012]
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- C_{0E} is upper bounded by the quantum Lovász number

$$C_{0E}(\mathcal{N}) \leq \log \widetilde{\vartheta}(\mathcal{N}).$$

  - Classical channel: [PRA, Beigi-2010]
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A pressing problem in zero-error information theory

- $C_{0E}(\mathcal{N}) = \log \widetilde{\vartheta}(\mathcal{N})$? (mentioned in the above articles)
- Open since 2010.
- If this is true, then C_{0E} is additive.

Difficulties of the problem and our approach

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Our approach

- Construct a class of qutrit-to-qutrit channel $\mathcal{N}_\alpha$
- Consider the zero-error communication properties of $\mathcal{N}_\alpha$ with the assistance of quantum no-signalling correlations (QNSC), which is a broader class of resources than entanglement.
- Compare $C_{0,NS}$ to quantum Lovász number

Construction of a class of qutrit-to-qutrit channel $\mathcal{N}_\alpha$

- The class of channels we use is

$$\mathcal{N}_\alpha(\rho) = E_\alpha \rho E_\alpha^\dagger + D_\alpha \rho D_\alpha^\dagger \ (0 < \alpha \leq \pi/4)$$

with

$$E_\alpha = \sin\alpha |0\rangle\langle 1| + |1\rangle\langle 2|,$$
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- The non-commutative graph of $\mathcal{N}_\alpha$ is $S = \text{span}\{F_1, F_2, F_3, F_4\}$ with

$$F_1 = |0\rangle\langle 0| + \cos^2\alpha |1\rangle\langle 1|,$$
$$F_2 = \sin^2\alpha |1\rangle\langle 1| + |2\rangle\langle 2|,$$
$$F_3 = |0\rangle\langle 2| \text{ and } F_4 = |2\rangle\langle 0|.$$

- The Choi-Jamiołkowski matrix of $\mathcal{N}_\alpha$ is given by

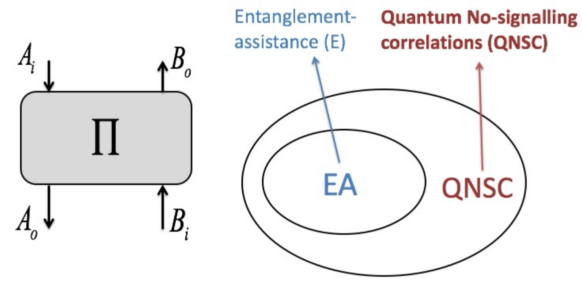
$$J_\alpha = (1 + \sin^2\alpha)|u\rangle\langle u| + (1 + \cos^2\alpha)|v\rangle\langle v|,$$

where

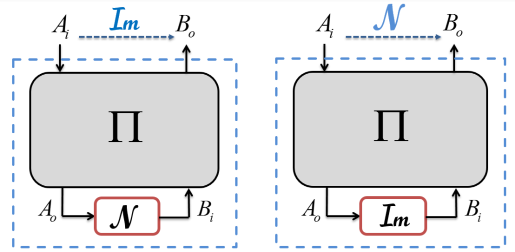
$$|u\rangle = \frac{\sin\alpha}{\sqrt{1+\sin^2\alpha}}|10\rangle + \frac{1}{\sqrt{1+\sin^2\alpha}}|21\rangle, |v\rangle = \frac{\cos\alpha}{\sqrt{1+\cos^2\alpha}}|12\rangle + \frac{1}{\sqrt{1+\cos^2\alpha}}|01\rangle.$$

Quantum no-signalling correlations (QNSC)

- Two-input and two-output quantum channels (bipartite CPTP linear map) $\Pi : \mathcal{L}(\mathcal{A}_i) \otimes \mathcal{L}(\mathcal{B}_i) \to \mathcal{L}(\mathcal{A}_o) \otimes \mathcal{L}(\mathcal{B}_o)$
- No-signalling correlations [Beckman-Gottesman-Nielsen-Preskill-01, Eggeling-Schlingemann-Werner-02, Piani-Horodecki et al.-06], A and B cannot use the channel to communicate classical information.
- The structure of QNSC is more mathematically tractable than the entanglement assistance.



QNSC-assisted zero-error capacity and simulation cost



- One-shot QNSC-assisted zero-error capacity is given by [Duan-Winter-2016]:

$$\Upsilon(\mathcal{N}) = \max \operatorname{Tr} S_A \ \text{ s.t. } \ 0 \leq U_{AB} \leq S_A \otimes \mathbb{1}_B, \quad \operatorname{Tr}_A U_{AB} = \mathbb{1}_B, \operatorname{Tr} J_{AB}(S_A \otimes \mathbb{1}_B - U_{AB}) = 0. \tag{1}$$

- Asymptotic: $C_{0,\mathrm{NS}}(\mathcal{N}) = \sup_{n\geq 1} \frac{1}{n} \log \Upsilon\left(\mathcal{N}^{\otimes n}\right)$

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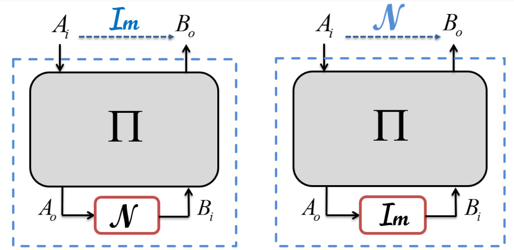

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- Asymptotic: $C_{0,\mathrm{NS}}(\mathcal{N}) = \sup_{n \ge 1} \frac{1}{n} \log \Upsilon\left(\mathcal{N}^{\otimes n}\right)$
- One-shot QNSC-assisted zero-error simulation cost [Duan-Winter-2016]:

$$\Sigma(\mathcal{N}) = 2^{-H_{\min}(A|B)_J} = \min \operatorname{Tr} T_B, \ \text{s.t. } J_{AB} \le \mathbb{1}_A \otimes T_B,$$

  where $H_{\min}(A|B)_J$ is the conditional min-entropy [Koenig-Renner-Schaffner-09].
- $\log \Upsilon(\mathcal{N}) \le C_{0,NS}(\mathcal{N}) \le C_E(\mathcal{N}) \le S_{0,NS}(\mathcal{N}) = \log \Sigma(\mathcal{N})$

QNSC-assisted zero-error capacity of $\mathcal{N}_\alpha$

For the channel $\mathcal{N}_\alpha$ $(0 < \alpha \leq \pi/4)$,

- $C_{0,NS}(\mathcal{N}_\alpha) \geq 2$
  - Construct a feasible solution to the prime SDP of $\Upsilon(\mathcal{N}_\alpha)$:

$$\begin{aligned} R_A =& 2(\cos^2\alpha|0\rangle\langle 0| + |1\rangle\langle 1| + \sin^2\alpha|2\rangle\langle 2|) \\ U_{AB} =& \cos^2\alpha|01\rangle\langle 01| + \sin^2|21\rangle\langle 21| + |10\rangle\langle 10| + |12\rangle\langle 12| \\ &+ \sin\alpha(|10\rangle\langle 21| + |21\rangle\langle 10|) + \cos\alpha(|01\rangle\langle 12| + |12\rangle\langle 01|). \end{aligned}$$

  - $C_{0,NS}(\mathcal{N}_\alpha) \geq \log\Upsilon(\mathcal{N}_\alpha) \geq \log \mathrm{Tr}\, R_A = 2$

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  - $C_{0,NS}(\mathcal{N}_\alpha) \geq \log\Upsilon(\mathcal{N}_\alpha) \geq \log \operatorname{Tr} R_A = 2$
- Similarly, we can prove that

$$S_{0,NS}(\mathcal{N}_\alpha) = -H_{\min}(A|B)_{J_{AB}} = \log\Sigma(\mathcal{N}) = 2.$$

- Recall $C_{0,NS}(\mathcal{N}) \leq C_E(\mathcal{N}) \leq S_{0,NS}(\mathcal{N})$

Proposition:

$$C_{0,NS}(\mathcal{N}_\alpha) = C_E(\mathcal{N}_\alpha) = S_{0,NS}(\mathcal{N}_\alpha) = 2.$$

Quantum Lovász number of $\mathcal{N}_\alpha$

Proposition: For the channel $\mathcal{N}_\alpha$ $(0 < \alpha \leq \pi/4)$,

$$\widetilde{\vartheta}(\mathcal{N}_\alpha) = 2 + \cos^2\alpha + \cos^{-2}\alpha > 4.$$

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Outline of proof:

- Recall the primal SDP of $\widetilde{\vartheta}(\mathcal{N}_\alpha)$:

$$\widetilde{\vartheta}(S) = \max \langle\Phi|(\mathbb{1}\otimes\rho + T)|\Phi\rangle$$
$$\text{s.t.} \quad T \in S^\perp \otimes \mathcal{L}(A'), \mathbb{1}\otimes\rho + T \geq 0, \operatorname{Tr}\rho = 1, \rho \geq 0.$$

- Suppose that $\rho = \frac{\cos^2\alpha}{1+\cos^2\alpha}|0\rangle\langle 0| + \frac{1}{1+\cos^2\alpha}|1\rangle\langle 1|$ and $T = T_1 \otimes T_2 + R$ with

$$T_1 = \frac{1}{1+\cos^2\alpha}\left(|0\rangle\langle 0| - \frac{1}{\cos^2\alpha}|1\rangle\langle 1| + \frac{\sin^2\alpha}{\cos^2\alpha}|2\rangle\langle 2|\right),$$
$$T_2 = \cos^4\alpha|0\rangle\langle 0| - |1\rangle\langle 1|, \; R = |00\rangle\langle 11| + |11\rangle\langle 00|.$$

- $\{\rho, T\}$ is a feasible solution to primal SDP of $\widetilde{\vartheta}(\mathcal{N}_\alpha)$. Hence,

$$\widetilde{\vartheta}(\mathcal{N}_\alpha) \geq \operatorname{Tr}[|\Phi\rangle\langle\Phi|(\mathbb{1}\otimes\rho + T)] = 2 + \cos^2\alpha + 1/\cos^2\alpha.$$

- Similarly, we can use the dual SDP to show

$$\widetilde{\vartheta}(\mathcal{N}_\alpha) \leq 2 + \cos^2\alpha + 1/\cos^2\alpha.$$

$C_{0E}(\mathcal{N}) \neq \log \widetilde{\vartheta}(\mathcal{N})$!!!

Theorem: For the channel $\mathcal{N}_\alpha$ $(0 < \alpha \leq \pi/4)$, the quantum Lovász number is strictly larger than the entanglement-assisted zero-error capacity (or even with quantum no-signalling assistance), i.e.,

$$\log \widetilde{\vartheta}(\mathcal{N}_\alpha) > C_{0,NS}(\mathcal{N}_\alpha) \geq C_{0E}(\mathcal{N}_\alpha).$$

- $C_{0E}(\mathcal{N}_\alpha) \leq C_{0,NS}(\mathcal{N}_\alpha) = C_E(\mathcal{N}_\alpha) = S_{0,NS}(\mathcal{N}_\alpha) = 2$
- $\log \widetilde{\vartheta}(\mathcal{N}_\alpha) = \log(2 + \cos^2 \alpha + \cos^{-2} \alpha) > 2$

Quantum fractional packing number and feedback-assisted or QNSC-assisted zero-error capacity

- For classical channel $\mathcal{N}$ [Shannon-1956, Cubitt-Leung-Matthews-Winter-2011],

$$C_{0F}(\mathcal{N}) = C_{0,NS}(\mathcal{N}) = \log \alpha^*(\mathcal{N}),$$

where $\alpha^*(\mathcal{N})$ is the fractional packing number:

$$\alpha^*(\mathcal{N}) = \max \sum_x v_x \ \text{ s.t. } \sum_x v_x \lceil \mathcal{N}(y|x) \rceil \leq 1 \ \forall y, 0 \leq v_x \leq 1 \ \forall x.$$

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- For classical-quantum channel $\mathcal{N}$ [Duan-Winter-2016],

$$C_{0,NS}(\mathcal{N}) = \log \mathrm{A}(\mathcal{N}),$$

where $\mathrm{A}(\mathcal{N})$ is the quantum fractional packing number:

$$\mathrm{A}(\mathcal{N}) = \max \operatorname{Tr} R_A \text{ s.t. } 0 \leq R_A, \operatorname{Tr}_A P_{AB}(R_A \otimes \mathbb{1}_B) \leq \mathbb{1}_B,$$

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- For classical-quantum channel $\mathcal{N}$ [Duan-Winter-2016],
$$C_{0,NS}(\mathcal{N}) = \log \mathrm{A}(\mathcal{N}),$$
where $\mathrm{A}(\mathcal{N})$ is the quantum fractional packing number:
$$\mathrm{A}(\mathcal{N}) = \max \operatorname{Tr} R_A \text{ s.t. } 0 \leq R_A, \operatorname{Tr}_A P_{AB}(R_A \otimes \mathbb{1}_B) \leq \mathbb{1}_B,$$
- However, for quantum channel $\mathcal{N}_\alpha$ $(0 < \alpha \leq \pi/4)$, we have that $\mathrm{A}(\mathcal{N}_\alpha) = 2 + \cos^2 \alpha + \cos^{-2} \alpha$ and
$$C_{0F}(\mathcal{N}_\alpha) < \log \mathrm{A}(\mathcal{N}_\alpha), C_{0,NS}(\mathcal{N}_\alpha) < \log \mathrm{A}(\mathcal{N}_\alpha).$$

Conclusion

Summary

- For the channel $\mathcal{N}_\alpha = E_\alpha \rho E_\alpha^\dagger + D_\alpha \rho D_\alpha^\dagger$ $(0 < \alpha \leq \pi/4)$ with

$$E_\alpha = \sin\alpha |0\rangle\langle 1| + |1\rangle\langle 2|, D_\alpha = \cos\alpha |2\rangle\langle 1| + |1\rangle\langle 0|,$$

  we have that

  - $C_{0,NS}(\mathcal{N}_\alpha) = C_E(\mathcal{N}_\alpha) = S_{0,NS}(\mathcal{N}_\alpha) = 2$ (A kind of reversibility)
  - $\widetilde{\vartheta}(\mathcal{N}_\alpha) = 2 + \cos^2\alpha + \cos^{-2}\alpha > 4$
  - $\mathsf{A}(K_\alpha) = 2 + \cos^2\alpha + \cos^{-2}\alpha > 4$

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- A separation between quantum Lovász number and entanglement-assisted zero-error classical capacity:

$$\log\widetilde{\vartheta}(\mathcal{N}_\alpha) > C_{0,NS}(\mathcal{N}_\alpha) \geq C_{0E}(\mathcal{N}_\alpha).$$

- A gap between quantum fractional packing number and feedback-assisted or QNSC-assisted zero-error capacity:

$$C_{0F}(\mathcal{N}_\alpha) < \log\mathrm{A}(\mathcal{N}_\alpha), C_{0,NS}(\mathcal{N}_\alpha) < \log\mathrm{A}(\mathcal{N}_\alpha).$$

Open Questions:

- It remains unknown whether Lovász number coincides with C_{0E} for every classical channel, i.e.,

$$C_{0E}(G) = \log \vartheta(G)?$$

- Other communication properties of $\mathcal{N}_\alpha$?
- Characterize the channels which have reversibility in the sense that

$$C_{0,NS}(\mathcal{N}) = S_{0,NS}(\mathcal{N}).$$

- Additivity for $C_{0E}(\mathcal{N})$ or $C_{0,NS}(\mathcal{N})$?

Thank you!
Questions?