Semidefinite programming strong converse bounds for classical capacity

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We investigate the classical communication over quantum channels when assisted by no-signalling (NS) and PPT-preserving (PPT) codes, for which both the optimal success probability of a given transmission rate and the one-shot ϵ-error capacity are formalized as semidefinite programs (SDPs). Based on this, we obtain improved SDP finite blocklength converse bounds of general quantum channels for entanglement-assisted codes and unassisted codes. Furthermore, we derive two SDP strong converse bounds for the classical capacity of general quantum channels: for any code with a rate exceeding either of the two bounds of the channel, the success probability vanishes exponentially fast as the number of channel uses increases. In particular, applying our efficiently computable bounds, we derive an improved upper bound on the classical capacity of the amplitude damping channel. We also establish the strong converse property for the classical and private capacities of a new class of quantum channels. We finally study the zero-error setting and provide efficiently computable upper bounds on the one-shot zero-error capacity of a general quantum channel.

I. INTRODUCTION

The reliable transmission of classical information via noisy quantum channels is central to quantum information theory. The classical capacity of a noisy quantum channel is the highest rate at which it can convey classical information reliably over asymptotically many uses of the channel. The Holevo-Schumacher-Westmoreland (HSW) theorem [2–4] gives a full characterization of the classical capacity of quantum channels:

$$C(\mathcal{N}) := \sup_{n\geq 1} \frac{\chi(\mathcal{N}^{\otimes n})}{n}, \tag{1}$$

where $\chi(\mathcal{N})$ is the Holevo capacity of the channel $\mathcal{N}$ given by $\chi(\mathcal{N}) := \max_{\{(p_i,\rho_i)\}} H\left(\sum_i p_i \mathcal{N}(\rho_i)\right) - \sum_i p_i H(\mathcal{N}(\rho_i))$, $\{(p_i,\rho_i)\}_i$ is an ensemble of quantum states on A and $H(\sigma) = -\operatorname{Tr}\sigma\log\sigma$ is the von Neumann entropy of a quantum state. Throughout this paper, $\log$ denotes the binary logarithm.

For certain classes of quantum channels (depolarizing channel [5], erasure channel [6], unital qubit channel [7], etc. [8–11]), the classical capacity of the channel is equal to the Holevo capacity, since their Holevo capacities are all additive. However, for a general quantum channel, our understanding of the classical capacity is still limited. The work of Hastings [12] shows that the

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Holevo capacity is generally not additive, thus the regularization in Eq. (1) is necessary in general. Since the complexity of computing the Holevo capacity is NP-complete [13], the regularized Holevo capacity of a general quantum channel is notoriously difficult to calculate. Even for the qubit amplitude damping channel, the classical capacity remains unknown.

The converse part of the HSW theorem states that if the communication rate exceeds the capacity, then the error probability of any coding scheme cannot approach zero in the limit of many channel uses. This kind of "weak" converse suggests the possibility for one to increase communication rates by allowing an increased error probability. A *strong converse property* leaves no such room for the trade-off, i.e., the error probability necessarily converges to one in the limit of many channel uses whenever the rate exceeds the capacity of the channel. For classical channels, the strong converse property for the classical capacity is established by Wolfowitz [14]. For quantum channels, the strong converse property for the classical capacity is confirmed for several classes of channels [15–19]. Winter [16] and Ogawa and Nagaoka [15] independently establish the strong converse property for the classical capacity of classical-quantum channels. Koenig and Wehner [17] prove the strong converse property for particular covariant quantum channels. Recently, for the entanglement-breaking and Hadamard channels, the strong converse property is proved by Wilde, Winter and Yang [19]. Moreover, the strong converse property for the pure-loss bosonic channel is proved by Wilde and Winter [18]. Unfortunately, for a general quantum channel, less is known about the strong converse property of the classical capacity and it remains open whether this property holds for all quantum channels. A *strong converse bound* for the classical capacity is a quantity such that the success probability of transmitting classical messages vanishes exponentially fast as the number of channel uses increases if the rate of communication exceeds this quantity, which forbids the trade-off between rate and error in the limit of many channel uses.

Another fundamental problem, of both theoretical and practical interest, is the trade-off between the channel uses, communication rate and error probability in the non-asymptotic (or finite blocklength) regime. In a realistic setting, the number of channel uses is necessarily limited in quantum information processing. Therefore one has to make a trade-off between the transmission rate and error tolerance. Note that one only needs to study one-shot communication over the channel since it can correspond to a finite blocklength and one can also study the asymptotic capacity via the finite blocklength approach. The study of finite blocklength regime has recently garnered great interest in classical information theory (e.g., [20–22]) as well as in quantum information theory (e.g., [23–35]). For classical channels, Polyanskiy, Poor, and Verdú [20] derive the finite blocklength converse bound via hypothesis testing and Matthews [22] provides an alternative proof of this converse bound via classical no-signalling codes. For classical-quantum channels, the one-shot converse and achievability bounds are given in [24, 25, 36]. Recently, the one-shot converse bounds for entanglement-assisted and unassisted codes were given in [23], which generalizes the hypothesis testing approach in [20] to quantum channels.

To gain insights into the generally intractable problem of evaluating the capacities of quantum channels, a natural approach is to study the performance of extra free resources in the coding scheme. This scheme, called a *code*, is equivalently a bipartite operation performed jointly by the sender Alice and the receiver Bob to assist the communication [28]. The *PPT-preserving codes*, i.e. the PPT-preserving bipartite operations, include all operations that can be implemented by local operations and classical communication (LOCC) and were introduced to study entanglement distillation in an early paper by Rains [37]. The *no-signalling (NS) codes* refer to the bipartite quantum operations with the no-signalling constraints, which arise in the research of the relativistic causality of quantum operations [38–41]. Recently these general codes have been used to study the zero-error classical communication [42] and quantum communication [28] over quantum channels. Our work follows this approach and focuses on classical communication via quantum channels assisted by NS and NS∩PPT codes.

II. SUMMARY OF RESULTS

In this paper, we focus on the reliable classical communication over quantum channels assisted by no-signalling and PPT-preserving codes under both non-asymptotic (or finite blocklength) and asymptotic settings. The summary of our results is as follows.

In Section IV, we formalize the optimal average success probability of transmitting classical messages over a quantum channel assisted by NS or NS∩PPT codes as SDPs. Using these SDPs, we establish the one-shot NS-assisted (or NS∩PPT-assisted) ϵ-error capacity, i.e., the maximum rate of classical communication with a fixed error threshold. We further compare these one-shot ϵ-error capacities with the previous SDP-computable entanglement-assisted (or unassisted) converse bound derived by the technique of quantum hypothesis testing in [23]. Our one-shot ϵ-error capacities, which consider potentially stronger assistances, are always no larger than the previous SDP bounds, and the inequalities can be strict even for qubit channels or classical-quantum channels. This means that our one-shot ϵ-error capacities can provide tighter finite blocklength converse bounds for the entanglement-assisted and unassisted classical capacity. Moreover, our one-shot ϵ-error capacities also reduce to the Polyanskiy-Poor-Verdú (PPV) converse bound [20] for classical channels. Furthermore, in common with the quantum hypothesis testing converse bound [23] and the bound of Datta and Hsieh [43], the large block length behaviour of our one-shot NS-assisted ϵ-error capacity also recovers the converse part of the formula for entanglement-assisted capacity [44] and implies that no-signalling-assisted classical capacity coincides with the entanglement-assisted classical capacity.

In Section V, we derive two SDP strong converse bounds for the NS∩PPT-assisted classical capacity of a general quantum channel based on the one-shot characterization of the optimal success probability. These bounds also provide efficiently computable strong converse bounds for the classical capacity. As a special case, we show that $\log(1+\sqrt{1-\gamma})$ is a strong converse bound for the classical capacity of the amplitude damping channel with parameter γ, and this improves the best previously known upper bound in [45]. Furthermore, applying our strong converse bounds, we also prove the strong converse property for the classical and private capacities of a new class of quantum channels.

In Section VI, we consider the zero-error communication problem [46], which requires that the communication is with zero probability of error. To be specific, based on our SDPs of optimal success probability, we derive the one-shot NS-assisted (or NS∩PPT-assisted) zero-error capacity of general quantum channels. Our result of the NS-assisted capacity provides an alternative proof of the NS-assisted zero-error capacity in [42]. Moreover, our one-shot NS∩PPT-assisted zero-error capacity gives an SDP-computable upper bound on the one-shot unassisted zero-error capacity, and it can be strictly smaller than the previous upper bound in [47].

Finally, in Section VII, we make a conclusion and leave some interesting open questions.

III. PRELIMINARIES

In the following, we will frequently use symbols such as A (or A') and B (or B') to denote (finite-dimensional) Hilbert spaces associated with Alice and Bob, respectively. We use d_A to denote the dimension of system A. The set of linear operators over A is denoted by $\mathcal{L}(A)$. We usually write an operator with subscript indicating the system that the operator acts on, such as T_{AB}, and write $T_A \coloneqq \operatorname{Tr}_B T_{AB}$. Note that for a linear operator $R \in \mathcal{L}(A)$, we define $|R| = \sqrt{R^\dagger R}$, where $R^\dagger$ is the adjoint operator of R, and the trace norm of R is given by $\|R\|_1 = \operatorname{Tr}|R|$. The operator norm $\|R\|_\infty$ is defined as the maximum eigenvalue of $|R|$. A deterministic quantum operation (quantum channel) $\mathcal{N}$ $(A' \to B)$ is simply a completely positive (CP) and trace-preserving

(TP) linear map from $\mathcal{L}(A')$ to $\mathcal{L}(B)$. The Choi-Jamiołkowski matrix [48, 49] of $\mathcal{N}$ is given by $J_\mathcal{N} = \sum_{ij} |i_A\rangle\langle j_A| \otimes \mathcal{N}(|i_{A'}\rangle\langle j_{A'}|)$, where $\{|i_A\rangle\}$ and $\{|i_{A'}\rangle\}$ are orthonormal bases on isomorphic Hilbert spaces A and A', respectively. A positive semidefinite operator $E \in \mathcal{L}(A \otimes B)$ is said to be a positive partial transpose operator (or simply PPT) if $E^{T_B} \geq 0$, where T_B means the partial transpose with respect to the party B, i.e., $(|ij\rangle\langle kl|)^{T_B} = |il\rangle\langle kj|$. As shown in [37], a bipartite operation $\Pi(A_iB_i \to A_oB_o)$ is PPT-preserving if and only if its Choi-Jamiołkowski matrix $Z_{A_iB_iA_oB_o}$ is PPT. We sometimes omit the identity operator or operation $\mathbb{1}$, for example, $\mathcal{E}(A \to B)(X_{AC}) \equiv (\mathcal{E}(A \to B) \otimes \mathbb{1}_C)(X_{AC})$.

The constraints of PPT and NS can be mathematically characterized as follows. A bipartite operation $\Pi(A_iB_i \to A_oB_o)$ is no-signalling and PPT-preserving if and only if its Choi-Jamiołkowski matrix $Z_{A_iB_iA_oB_o}$ satisfies [28]:

$$\begin{aligned}
Z_{A_iB_iA_oB_o} &\geq 0, \quad &(\text{CP})\\
Z_{A_iB_i} &= \mathbb{1}_{A_iB_i}, \quad &(\text{TP})\\
Z^{T_{B_iB_o}}_{A_iB_iA_oB_o} &\geq 0, \quad &(\text{PPT})\\
Z_{A_iB_iB_o} &= \frac{\mathbb{1}_{A_i}}{d_{A_i}} \otimes Z_{B_iB_o}, \quad &(A \not\to B)\\
Z_{A_iB_iA_o} &= \frac{\mathbb{1}_{B_i}}{d_{B_i}} \otimes Z_{A_iA_o}, \quad &(B \not\to A)
\end{aligned} \tag{2}$$

where the five lines correspond to characterize that Π is completely positive, trace-preserving, PPT-preserving, no-signalling from A to B, no-signalling from B to A, respectively. The structure of no-signalling codes is also studied in [42].

Semidefinite programming [50] is a subfield of convex optimization and is a powerful tool in quantum information theory with many applications (e.g., [23, 28, 37, 42, 51–56]). There are known polynomial-time algorithms for semidefinite programming [57]. In this work, we use the CVX software (a Matlab-based convex modeling framework) [58] and QETLAB (A Matlab Toolbox for Quantum Entanglement) [59] to solve the SDPs. Details about semidefinite programming can be found in [60].

IV. CLASSICAL COMMUNICATION ASSISTED BY NS AND PPT CODES

A. Semidefinite programs for optimal success probability

Suppose Alice wants to send the classical message labeled by $\{1, \ldots, m\}$ to Bob using the composite channel $\mathcal{M} = \Pi \circ \mathcal{N}$, where Π is a bipartite operation that generalizes the usual encoding scheme $\mathcal{E}$ and decoding scheme $\mathcal{D}$, see Fig. 1 for details. In this paper, we consider Π as the bipartite operation implementing the NS $\cap$ PPT or NS assistance. After the action of $\mathcal{E}$ and $\mathcal{N}$, the message results in quantum state at Bob's side. Bob then performs a POVM with m outcomes on the resulting quantum state. The POVM is a component of the operation $\mathcal{D}$. Since the results of the POVM and the input messages are both classical, it is natural to assume that $\mathcal{M}$ is with classical registers throughout this paper, that is, $\Delta \circ \mathcal{M} \circ \Delta = \mathcal{M}$ for some completely dephasing channel Δ. If the outcome $k \in \{1, \ldots, m\}$ happens, he concludes that the message with label k was sent. Let Ω be some class of bipartite operations. The average success probability of the general code Π and the Ω-class code is defined as follows.

Definition 1 *The average success probability of $\mathcal{N}$ to transmit m messages assisted with the code Π is*

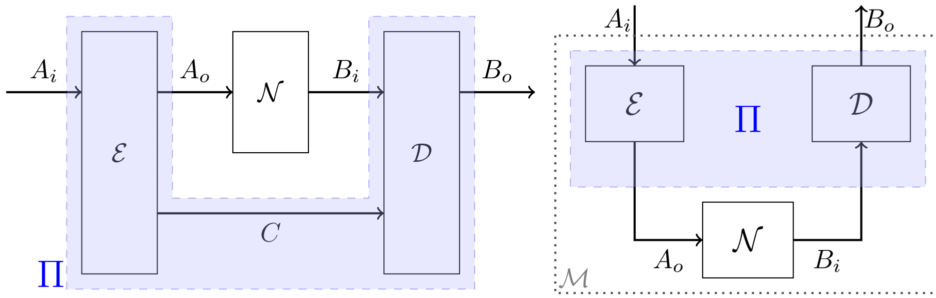

FIG. 1: Bipartite operation $\Pi(A_iB_i \to A_oB_o)$ is equivalently the coding scheme $(\mathcal{E},\mathcal{D})$ with free extra resources, such entanglement or no-signalling correlations. The whole operation is to emulate a noiseless classical (or quantum) channel $\mathcal{M}(A_i \to B_o)$ using a given noisy quantum channel $\mathcal{N}(A_o \to B_i)$ and the bipartite operation Π.

defined by

$$f(\mathcal{N}, \Pi, m) = \frac{1}{m}\sum_{k=1}^{m} \operatorname{Tr}(\mathcal{M}(|k\rangle\langle k|)|k\rangle\langle k|), \tag{3}$$

where $\mathcal{M} \equiv \Pi \circ \mathcal{N}$ and $\{|k\rangle\}$ is the computational basis in system A_i.

Furthermore, the optimal average success probability of $\mathcal{N}$ to transmit m messages assisted with Ω-class code is defined by

$$f_\Omega(\mathcal{N}, m) = \sup_{\Pi} f(\mathcal{N}, \Pi, m), \tag{4}$$

where the maximum is over the codes in class Ω.

We now define the Ω-assisted classical capacity of a quantum channel as follows.

Definition 2

$$C_\Omega(\mathcal{N}) := \sup\left\{ r : \lim_{n\to\infty} f_\Omega(\mathcal{N}^{\otimes n}, 2^{rn}) = 1 \right\}. \tag{5}$$

As described above, one can simulate a channel $\mathcal{M}$ with the channel $\mathcal{N}$ and code Π, where Π is a bipartite CPTP operation from A_iB_i to A_oB_o which is no-signalling (NS) and PPT-preserving (PPT). In this work we shall also consider other classes of codes, such as entanglement-assisted (EA) code, unassisted (UA) code. The class of entanglement-assisted codes corresponds to bipartite operations of the form $\Pi(A_iB_i \to A_oB_o) = \mathcal{D}(B_i\hat{B} \to B_o)\mathcal{E}(A_i\hat{A} \to A_o)\varphi_{\hat{A}\hat{B}}$, where $\mathcal{E}, \mathcal{D}$ are encoding and decoding operations respectively, and $\varphi_{\hat{A}\hat{B}}$ can be any shared entangled state of arbitrary systems $\hat{A}$ and $\hat{B}$. we use Ω to denote specific class of codes such as $\Omega \in \{\mathrm{NS}, \mathrm{PPT}, \mathrm{NS} \cap \mathrm{PPT}, \mathrm{EA}, \mathrm{UA}\}$ in the following.

Let $\mathcal{M}(A_i \to B_o)$ denote the resulting composition channel of Π and $\mathcal{N}$, written $\mathcal{M} = \Pi \circ \mathcal{N}$. As both $\mathcal{M}$ and $\mathcal{N}$ are quantum channels, there exist quantum channels $\mathcal{E}(A_i \to A_oC)$ and $\mathcal{D}(B_iC \to B_o)$, where $\mathcal{E}$ is an isometry operation and C is a quantum register, such that [61]

$$\mathcal{M}(A_i \to B_o) = \mathcal{D}(B_iC \to B_0) \circ \mathcal{N}(A_o \to B_i) \circ \mathcal{E}(A_i \to A_oC). \tag{6}$$

Based on this, the Choi-Jamiołkowski matrix of $\mathcal{M}$ is given by [28]

$$J_{\mathcal{M}} = \operatorname{Tr}_{A_oB_i}(J_{\mathcal{N}}^T \otimes \mathbb{1}_{A_iB_o})Z_{A_iA_oB_iB_o}. \tag{7}$$

The operations $\mathcal{E}$ and $\mathcal{D}$ can be considered as generalized encoding and decoding operations respectively, except that the register C may be not possessed by Alice or Bob. If the Hilbert space with C is trivial, $\mathcal{E}$ and $\mathcal{D}$ become the unassisted local encoding/decoding operations. Moreover, the coding schemes $\mathcal{E}, \mathcal{D}$ with register C can be designed to be forward-assisted codes [28].

We are now able to derive the one-shot characterization of classical communication assisted by NS (or NS∩PPT) codes.

Theorem 3 *For a given quantum channel $\mathcal{N}$, the optimal success probability of $\mathcal{N}$ to transmit m messages assisted by NS∩PPT codes is given by*

$$\begin{aligned} f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}, m) = \max\ & \operatorname{Tr} J_{\mathcal{N}} F_{AB} \\ \text{s.t. } & 0 \le F_{AB} \le \rho_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr} \rho_A = 1, \\ & \operatorname{Tr}_A F_{AB} = \mathbb{1}_B / m, \\ & 0 \le F_{AB}^{T_B} \le \rho_A \otimes \mathbb{1}_B \ (\mathrm{PPT}). \end{aligned} \tag{8}$$

Similarly, when assisted by NS codes, one can remove the PPT constraint to obtain the optimal success probability as follows:

$$\begin{aligned} f_{\mathrm{NS}}(\mathcal{N}, m) = \max\ & \operatorname{Tr} J_{\mathcal{N}} F_{AB} \\ \text{s.t. } & 0 \le F_{AB} \le \rho_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr} \rho_A = 1, \\ & \operatorname{Tr}_A F_{AB} = \mathbb{1}_B / m. \end{aligned} \tag{9}$$

Proof In this proof, we first use the Choi-Jamiołkowski representations of quantum channels to refine the average success probability and then exploit symmetry to simplify the optimization over all possible codes. Finally, we impose the no-signalling and PPT-preserving constraints to obtain the semidefinite program of the optimal average success probability.

Without loss of generality, we assume that A_i and B_o are classical registers with size m, i.e., the inputs and outputs are $\{|k\rangle_{A_i}\}_{k=1}^m$ and $\{|k'\rangle_{B_i}\}_{k'=1}^m$, respectively. For some NS∩PPT code Π, the Choi-Jamiołkowski matrix of $\mathcal{M} = \Pi \circ \mathcal{N}$ is given by $J_{\mathcal{M}} = \sum_{ij} |i\rangle\langle j|_{A_i} \otimes \mathcal{M}(|i\rangle\langle j|_{A_i'})$, where A_i' is isometric to A_i. Then, we can simplify $f(\mathcal{N}, \Pi, m)$ to

$$\begin{aligned} & f(\mathcal{N}, \Pi, m) \\ & = \frac{1}{m} \sum_{k=1}^m \operatorname{Tr}\left(\mathcal{M}(|k\rangle\langle k|_{A_i'}) |k\rangle\langle k|_{B_o}\right) \\ & = \frac{1}{m} \operatorname{Tr}\left(\sum_{i,j=1}^m (|i\rangle\langle j|_{A_i} \otimes \mathcal{M}(|i\rangle\langle j|_{A_i'})) \sum_{k=1}^m |kk\rangle\langle kk|_{A_i B_o}\right) \\ & = \frac{1}{m} \operatorname{Tr} J_{\mathcal{M}} \sum_{k=1}^m |kk\rangle\langle kk|_{A_i B_o}. \end{aligned} \tag{10}$$

Then, denoting $D_{A_i B_o} = \sum_{k=1}^m |kk\rangle\langle kk|_{A_i B_o}$, we have

$$f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}, m) = \max_{\mathcal{M} = \Pi \circ \mathcal{N}} \frac{1}{m} \operatorname{Tr}(J_{\mathcal{M}} D_{A_i B_o}),$$

where $\mathcal{M} = \Pi \circ \mathcal{N}$ and Π is any feasible NS∩PPT bipartite operation . (See FIG. 1 for the implementation of $\mathcal{M}$.) Noting that $J_{\mathcal{M}} = \operatorname{Tr}_{A_o B_i}(J_{\mathcal{N}}^T \otimes \mathbb{1}_{A_i B_o}) Z_{A_i A_o B_i B_o}$, we can further simplify $f(\mathcal{N}, m)$

as

$$
\begin{aligned}
&f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N},m)\\
&\ = \max\ \operatorname{Tr}(J_{\mathcal{N}}^{T}\otimes \mathbb{1}_{A_iB_o})Z_{A_iA_oB_iB_o}(\mathbb{1}_{A_oB_i}\otimes D_{A_iB_o})/m,\\
&\quad\ \ \text{s. t. } Z_{A_iA_oB_iB_o} \text{ satisfies Eq. (2).}
\end{aligned}
\tag{11}
$$

The next step is to simplify $f(\mathcal{N},m)$ by exploiting symmetry. For any permutation $\tau\in S_m$, where S_m is the symmetric group of degree m, if $Z_{A_iA_oB_iB_o}$ is feasible (satisfying the constraints in Eq. (2)), then it is not difficult to check that

$$
Z'_{A_iA_oB_iB_o} = (\tau_{A_i}\otimes\tau_{B_o}\otimes\mathbb{1}_{A_oB_i})Z_{A_iA_oB_iB_o}(\tau_{A_i}\otimes\tau_{B_o}\otimes\mathbb{1}_{A_oB_i})^{\dagger}
\tag{12}
$$

is also feasible. And any convex combination $\lambda Z'+(1-\lambda)Z''(0\le\lambda\le 1)$ of two operators satisfying Eq. (2) can also checked to be feasible. Therefore, if $Z_{A_iA_oB_iB_o}$ is feasible, so is

$$
\begin{aligned}
\widetilde{Z}_{A_iA_oB_iB_o} &= \mathcal{P}_{A_iB_o}(Z_{A_iA_oB_iB_o})\\
&:= \frac{1}{m!}\sum_{\tau_{A_i},\tau_{B_o}\in S_m}(\tau_{A_i}\otimes\tau_{B_o})Z_{A_iA_oB_iB_o}(\tau_{A_i}\otimes\tau_{B_o})^{\dagger},
\end{aligned}
\tag{13}
$$

where $\mathcal{P}_{A_iB_o}$ is a twirling operation on A_iB_o.

Noticing that $\mathcal{P}_{A_iB_o}(D_{A_iB_o}) = D_{A_iB_o}$, we have

$$
\begin{aligned}
&\operatorname{Tr}_{A_iB_o}(Z_{A_iB_iA_oB_o}(\mathbb{1}_{A_oB_i}\otimes D_{A_iB_o}))\\
&\qquad = \operatorname{Tr}_{A_iB_o}(Z_{A_iB_iA_oB_o}(\mathbb{1}_{A_oB_i}\otimes \mathcal{P}_{A_iB_o}(D_{A_iB_o}))\\
&\qquad = \operatorname{Tr}_{A_iB_o}(\widetilde{Z}_{A_iA_oB_iB_o}(\mathbb{1}_{A_oB_i}\otimes D_{A_iB_o})).
\end{aligned}
\tag{14}
$$

Thus, it is easy to see that the optimal success probability equals to

$$
\begin{aligned}
&f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N},m)\\
&\ = \max\ \operatorname{Tr}(J_{\mathcal{N}}^{T}\otimes \mathbb{1}_{A_iB_o})\widetilde{Z}_{A_iA_oB_iB_o}(\mathbb{1}_{A_oB_i}\otimes D_{A_iB_o})/m\\
&\quad\ \ \text{s. t. } \widetilde{Z}_{A_iA_oB_iB_o} \text{ satisfies Eq. (2).}
\end{aligned}
$$

It is worth noting that $\widetilde{Z}_{A_iA_oB_iB_o}$ can be rewritten as [42]

$$
\widetilde{Z}_{A_iA_oB_iB_o} = F_{A_oB_i}\otimes D_{A_iB_o} + E_{A_oB_i}\otimes(\mathbb{1}-D_{A_iB_o}),
$$

for some operators $E_{A_oB_i}$ and $F_{A_oB_i}$. Thus, the objective function can be simplified to $\operatorname{Tr} J_{\mathcal{N}}^{T}F$. Also, the CP and PPT constraints are equivalent to

$$
E_{A_oB_i}\ge 0, F_{A_oB_i}\ge 0, E_{A_oB_i}^{T_{B_i}}\ge 0, F_{A_oB_i}^{T_{B_i}}\ge 0.
\tag{15}
$$

Furthermore, the $B\not\to A$ constraint is equivalent to $\operatorname{Tr}_{B_o}\widetilde{Z}_{A_iA_oB_iB_o} = \operatorname{Tr}_{B_oB_i}\widetilde{Z}_{A_iA_oB_iB_o}\otimes\mathbb{1}_{B_i}/d_{B_i}$, i.e.

$$
\begin{aligned}
&F_{A_oB_i}+(m-1)E_{A_oB_i}\\
&= \operatorname{Tr}_{B_i}(F_{A_oB_i}+(m-1)E_{A_oB_i})\otimes\frac{\mathbb{1}_{B_i}}{d_{B_i}} =: \rho_{A_o}\otimes\mathbb{1}_{B_i}.
\end{aligned}
\tag{16}
$$

and the TP constraint holds if and only if $\operatorname{Tr}_{A_oB_o} Z_{A_iA_oB_iB_o} = \mathbb{1}_{A_iB_i}$, i.e.,

$$
\operatorname{Tr}_{A_o}(F_{A_oB_i}+(m-1)E_{A_oB_i}) = \mathbb{1}_{B_i},
\tag{17}
$$

which is equivalent to

$$\operatorname{Tr}\rho_{A_o} = \operatorname{Tr}(F_{A_oB_i} + (m-1)E_{A_oB_i})/d_{B_i} = \operatorname{Tr}\mathbb{1}_{B_i}/d_{B_i} = 1. \tag{18}$$

As Π is no-signalling from A to B, we have $\operatorname{Tr}_{A_o}\widetilde{Z}_{A_iA_oB_iB_o} = \operatorname{Tr}_{A_oA_i}\widetilde{Z}_{A_iA_oB_iB_o}\otimes\frac{\mathbb{1}_{A_i}}{m}$, i.e.,

$$\begin{aligned}&\operatorname{Tr}_{A_o}F_{A_oB_i}\otimes D_{A_iB_o} + \operatorname{Tr}_{A_o}E_{A_oB_i}\otimes(\mathbb{1}-D_{A_iB_o})\\ &= \operatorname{Tr}_{A_o}(F_{A_oB_i}+(m-1)E_{A_oB_i})\otimes\frac{\mathbb{1}_{A_iB_o}}{m} = \mathbb{1}_{A_iB_iB_o}/m.\end{aligned} \tag{19}$$

Since $D_{A_iB_o}$ and $\mathbb{1}-D_{A_iB_o}$ are orthogonal positive operators, we have

$$\operatorname{Tr}_{A_o}F_{A_oB_i} = \operatorname{Tr}_{A_o}E_{A_oB_i} = \mathbb{1}_{B_i}/m. \tag{20}$$

Finally, combining Eq. (15), (16), (18), (20), we have that

$$\begin{aligned}f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N},m) = \max\ &\operatorname{Tr}J_{\mathcal{N}}F_{A_oB_i}\\ \text{s.t. } &0\le F_{A_oB_i}\le\rho_{A_o}\otimes\mathbb{1}_{B_i},\\ &\operatorname{Tr}\rho_{A_o}=1,\\ &\operatorname{Tr}_{A_o}F_{A_oB_i}=\mathbb{1}_{B_i}/m,\\ &0\le F_{A_oB_i}^{T_{B_i}}\le\rho_{A_o}\otimes\mathbb{1}_{B_i}.\end{aligned} \tag{21}$$

This gives the SDP in Theorem 3, where we assume that $A_o = A$ and $B_i = B$ for simplification. □

Remark: The dual SDP for $f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N},m)$ is given by

$$\begin{aligned}f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N},m) = \min\ &t+\operatorname{Tr}S_B/m\\ \text{s.t.}\ &J_{\mathcal{N}}\le X_{AB}+\mathbb{1}_A\otimes S_B+(W_{AB}-Y_{AB})^{T_B},\\ &\operatorname{Tr}_B(X_{AB}+W_{AB})\le t\mathbb{1}_A,\\ &X_{AB},Y_{AB},W_{AB}\ge 0, S_B=S_B^\dagger.\end{aligned} \tag{22}$$

To remove the PPT constraint, set $Y_{AB} = W_{AB} = 0$. It is worth noting that the strong duality holds here since the Slater's condition can be easily checked. Indeed, choosing $X_{AB} = Y_{AB} = W_{AB} = \|J_{\mathcal{N}}\|_\infty\mathbb{1}_{AB}$, $S_B = \mathbb{1}_B$ and $t = 3d_B\|J_{\mathcal{N}}\|_\infty$ in SDP (22), we have $(X_{AB}, Y_{AB}, W_{AB}, S_B, t)$ is in the relative interior of the feasible region.

It is worthing noting that $f_{\mathrm{NS}}(\mathcal{N},m)$ can be obtained by removing the PPT constraint and it corresponds with the optimal NS-assisted channel fidelity in [28].

B. Improved SDP converse bounds in finite blocklength

For given $0\le\epsilon<1$, the *one-shot ϵ-error classical capacity assisted by Ω-class codes* is defined as

$$C_\Omega^{(1)}(\mathcal{N},\epsilon) := \sup\{\log\lambda : 1-f_\Omega(\mathcal{N},\lambda)\le\epsilon\}. \tag{23}$$

We now derive the one-shot ϵ-error classical capacity assisted by NS or NS∩PPT codes as follows.

Theorem 4 *For given channel $\mathcal{N}$ and error threshold ϵ, the one-shot ϵ-error* NS∩PPT-assisted and NS-assisted capacities are given by

$$C^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\epsilon) = -\log\min\ \eta \\ \text{s.t. } 0 \le F_{AB} \le \rho_A \otimes \mathbb{1}_B, \\ \operatorname{Tr}\rho_A = 1, \operatorname{Tr}_A F_{AB} = \eta \mathbb{1}_B, \\ \operatorname{Tr} J_{\mathcal{N}} F_{AB} \ge 1-\epsilon, \\ 0 \le F_{AB}^{T_B} \le \rho_A \otimes \mathbb{1}_B \text{ (PPT)}, \tag{24}$$

and

$$C^{(1)}_{\text{NS}}(\mathcal{N},\epsilon) = -\log\min\ \eta \\ \text{s.t. } 0 \le F_{AB} \le \rho_A \otimes \mathbb{1}_B, \operatorname{Tr}\rho_A = 1, \\ \operatorname{Tr}_A F_{AB} = \eta \mathbb{1}_B, \operatorname{Tr} J_{\mathcal{N}} F_{AB} \ge 1-\epsilon, \tag{25}$$

respectively.

Proof When assisted by NS∩PPT codes, by Eq. (23), we have that

$$C^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\epsilon) = \log\max\lambda \text{ s.t.} \mathrm{f}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\lambda) \ge 1-\epsilon. \tag{26}$$

To simplify Eq. (26), we suppose that

$$\Upsilon(\mathcal{N},\epsilon) = -\log\min\ \eta \\ \text{s.t. } 0 \le F_{AB} \le \rho_A \otimes \mathbb{1}_B, \\ \operatorname{Tr}\rho_A = 1, \operatorname{Tr}_A F_{AB} = \eta \mathbb{1}_B, \\ \operatorname{Tr} J_{\mathcal{N}} F_{AB} \ge 1-\epsilon, \\ 0 \le F_{AB}^{T_B} \le \rho_A \otimes \mathbb{1}_B \text{ (PPT)}. \tag{27}$$

On one hand, for given ϵ, suppose that the optimal solution to the SDP (27) of $\Upsilon(\mathcal{N},\epsilon)$ is $\{\rho, F, \eta\}$. Then, it is clear that $\{\rho, F\}$ is a feasible solution of the SDP (8) of $f_{\text{NS}\cap\text{PPT}}(\mathcal{N},\eta^{-1})$, which means that $f_{\text{NS}\cap\text{PPT}}(\mathcal{N},\eta^{-1}) \ge \operatorname{Tr} J_{\mathcal{N}} F \ge 1-\epsilon$. Therefore,

$$C^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\epsilon) \ge \log\eta^{-1} = \Upsilon(\mathcal{N},\epsilon). \tag{28}$$

On the other hand, for given ϵ, suppose that the value of $C^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\epsilon)$ is $\log\lambda$ and the optimal solution of $f_{\text{NS}\cap\text{PPT}}(\mathcal{N},\lambda)$ is $\{\rho, F\}$. It is easy to check that $\{\rho, F, \lambda^{-1}\}$ satisfies the constrains in SDP (27) of $\Upsilon(\mathcal{N},\epsilon)$. Therefore,

$$\Upsilon(\mathcal{N},\epsilon) \ge -\log\lambda^{-1} = C^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\epsilon). \tag{29}$$

Hence, combining Eqs. (27), (28) and (29), it is clear that

$$C^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\epsilon) = \Upsilon(\mathcal{N},\epsilon) \\ = -\log\min\eta \\ \text{s.t. } 0 \le F_{AB} \le \rho_A \otimes \mathbb{1}_B, \\ \operatorname{Tr}\rho_A = 1, \operatorname{Tr}_A F_{AB} = \eta \mathbb{1}_B, \\ \operatorname{Tr} J_{\mathcal{N}} F_{AB} \ge 1-\epsilon, \\ 0 \le F_{AB}^{T_B} \le \rho_A \otimes \mathbb{1}_B \text{ (PPT)}. \tag{30}$$

And one can obtain $C^{(1)}_{\rm NS}(\mathcal{N},\epsilon)$ by removing the PPT constraint. □

Noticing that no-signalling-assisted codes are potentially stronger than the entanglement-assisted codes, $C^{(1)}_{\rm NS}(\mathcal{N},\epsilon)$ and $C^{(1)}_{\rm NS\cap PPT}(\mathcal{N},\epsilon)$ provide converse bounds of classical communication for entanglement-assisted and unassisted codes, respectively.

Corollary 5 *For a given channel $\mathcal{N}$ and error threshold ϵ,*

$$
\begin{aligned}
&C^{(1)}_{\rm E}(\mathcal{N},\epsilon) \le C^{(1)}_{\rm NS}(\mathcal{N},\epsilon),\\
&C^{(1)}(\mathcal{N},\epsilon) \le C^{(1)}_{\rm NS\cap PPT}(\mathcal{N},\epsilon).
\end{aligned}
$$

We further compare our one-shot ϵ-error capacities with the previous SDP converse bounds derived by the quantum hypothesis testing technique in [23]. To be specific, for a given channel $\mathcal{N}(A\to B)$ and error thresold ϵ, Matthews and Wehner [23] establish that

$$
\begin{aligned}
C^{(1)}_{\rm E}(\mathcal{N},\epsilon) &\le R_{\rm E}(\mathcal{N},\epsilon)\\
&= \max_{\rho_A}\min_{\sigma_B} D^{\epsilon}_H((id_{A'}\otimes\mathcal{N})(\rho_{A'A})\|\rho_{A'}\otimes\sigma_B)\\
&= -\log\min\eta\\
&\qquad \text{s.t.} 0\le \mathrm{F_{AB}} \le \rho_{\rm A}\otimes \mathbb{1}_{\rm B}, \mathrm{Tr}\,\rho_{\rm A}=1,\\
&\qquad\qquad \mathrm{Tr}_A F_{AB}\le \eta \mathbb{1}_B, \mathrm{Tr}\, J_{\mathcal{N}} F_{AB}\ge 1-\epsilon,
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
C^{(1)}(\mathcal{N},\epsilon) &\le R_{\rm E\cap PPT}(\mathcal{N},\epsilon)\\
&= \max_{\rho_A}\min_{\sigma_B} D^{\epsilon}_{H,PPT}((id_{A'}\otimes\mathcal{N})(\rho_{A'A})\|\rho_{A'}\otimes\sigma_B)\\
&= -\log\min\eta\\
&\qquad \text{s.t. } 0\le F_{AB}\le \rho_A\otimes\mathbb{1}_B, \mathrm{Tr}\,\rho_A=1,\\
&\qquad\qquad \mathrm{Tr}_A F_{AB}\le\eta\mathbb{1}_B, \mathrm{Tr}\, J_{\mathcal{N}}F_{AB}\ge 1-\epsilon,\\
&\qquad\qquad 0\le F_{AB}^{T_B}\le \rho_A\otimes\mathbb{1}_B,
\end{aligned}
\tag{32}
$$

where $\rho_{A'A}=(\mathbb{1}_{A'}\otimes\rho_A^{\frac12})\Phi_{A'A}(\mathbb{1}_{A'}\otimes\rho_A^{\frac12})$ is a purification of ρ_A and $\rho_{A'}=\mathrm{Tr}_A\,\rho_{A'A}$. Moreover,

$$
\begin{aligned}
D^{\epsilon}_H(\rho_0\|\rho_1) = -\log\min\ &\mathrm{Tr}\,T\rho_1\\
&\text{s.t. } 1-\mathrm{Tr}\,T\rho_0\le\epsilon, 0\le T\le\mathbb{1}
\end{aligned}
\tag{33}
$$

is the hypothesis testing relative entropy [23, 24] and $D^{\epsilon}_{H,PPT}(\rho_0\|\rho_1)$ is the similar quantity with a PPT constraint on the POVM.

Interestingly, our one-shot ϵ-error capacities are similar to these quantum hypothesis testing relative entropy converse bounds. However, there is a crucial difference that our quantities require that a stricter condition, i.e., $\mathrm{Tr}_A F_{AB}=\eta\mathbb{1}_B$. This makes one-shot ϵ-error capacities ($C^{(1)}_{\rm NS\cap PPT}(\mathcal{N},\epsilon)$ and $C^{(1)}_{\rm NS}(\mathcal{N},\epsilon)$) always smaller than or equal to the SDP converse bounds in [23], and the inequalities can be strict.

Proposition 6 *For a given channel $\mathcal{N}(A\to B)$ and error threshold ϵ,*

$$
\begin{aligned}
C^{(1)}_{\rm NS}(\mathcal{N},\epsilon) &\le R_{\rm E}(\mathcal{N},\epsilon)\\
&= \max_{\rho_A}\min_{\sigma_B} D^{\epsilon}_H((id_{A'}\otimes\mathcal{N})(\rho_{A'A})\|\rho_{A'}\otimes\sigma_B),\\
C^{(1)}_{\rm NS\cap PPT}(\mathcal{N},\epsilon) &\le R_{\rm E\cap PPT}(\mathcal{N},\epsilon)\\
&= \max_{\rho_A}\min_{\sigma_B} D^{\epsilon}_{H,PPT}((id_{A'}\otimes\mathcal{N})(\rho_{A'A})\|\rho_{A'}\otimes\sigma_B).
\end{aligned}
$$

In particular, both inequalities can be strict for some quantum channels such as the amplitude damping channels and the simplest classical-quantum channels.

Proof This can be proved by the fact that any feasible solution of the SDP (25) of $C_{\mathrm{NS}}^{(1)}(\mathcal{N},\epsilon)$ (or $C_{\mathrm{NS}\cap\mathrm{PPT}}^{(1)}(\mathcal{N},\epsilon)$) is also feasible to the SDP (31) of $R_{\mathrm{E}}(\mathcal{N},\epsilon)$ (or $R_{\mathrm{E}\cap\mathrm{PPT}}(\mathcal{N},\epsilon)$).

We further show that the inequality can be strict by the example of qubit amplitude damping channel $\mathcal{N}_\gamma^{AD} = \sum_{i=0}^{1} E_i \cdot E_i^\dagger$ $(0 \le \gamma \le 1)$, with $E_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$ and $E_1 = \sqrt{\gamma}|0\rangle\langle 1|$. We compare the above bounds in FIG. 2 and FIG. 3. It is clear that our bounds can be strictly better than the quantum hypothesis testing bounds in [23] in this case.

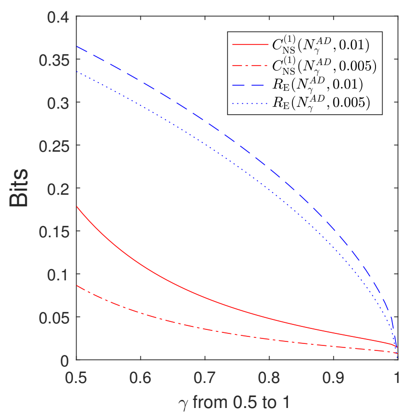

FIG. 2: The red solid and dash-dot lines depict $C_{\mathrm{NS}}^{(1)}(\mathcal{N}_\gamma^{AD},0.01)$ and $C_{\mathrm{NS}}^{(1)}(\mathcal{N}_\gamma^{AD},0.005)$, respectively. The blue dashed and dotted lines depict $R_{\mathrm{E}}(\mathcal{N}_\gamma^{AD},0.01)$ and $R_{\mathrm{E}}(\mathcal{N}_\gamma^{AD},0.005)$.

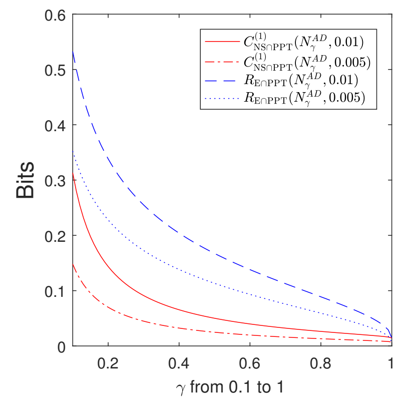

FIG. 3: The red solid and dash-dot lines depict $C_{\mathrm{NS}\cap\mathrm{PPT}}^{(1)}(\mathcal{N}_\gamma^{AD},0.01)$ and $C_{\mathrm{NS}\cap\mathrm{PPT}}^{(1)}(\mathcal{N}_\gamma^{AD},0.005)$, respectively. The blue dashed and dotted lines depict $R_{\mathrm{E}\cap\mathrm{PPT}}(\mathcal{N}_\gamma^{AD},0.01)$ and $R_{\mathrm{E}\cap\mathrm{PPT}}(\mathcal{N}_\gamma^{AD},0.005)$, respectively.

Another example is the simplest classical-quantum channel $\mathcal{N}_a^{cq}$ which has only two inputs and two pure output states $|\psi_i\rangle\langle\psi_i|$, w.l.o.g.

$$|\psi_0\rangle = a|0\rangle + b|1\rangle,$$
$$|\psi_1\rangle = a|0\rangle - b|1\rangle,$$

with $a \ge b = \sqrt{1-a^2}$. The comparison is presented in FIG. 4 and it is clear that our bound can be strictly tighter for this class of classical-quantum channels.

□

We then consider the asymptotic performance of the one-shot NS-assisted ϵ-error capacity. Interestingly, in common with the bound $R_{\mathrm{E}}(\mathcal{N},\epsilon)$ [23] and the bound of Datta and Hsieh [43], the asymptotic behaviour of $R_{\mathrm{NS}}(\mathcal{N},\epsilon)$ also recovers the converse part of the formula for entanglement-assisted capacity [44] and it implies that $C_{\mathrm{NS}}(\mathcal{N}) = C_{\mathrm{E}}(\mathcal{N})$. (See Corollary 7.) In [28], Leung and Matthews have already shown that the entanglement-assisted quantum capacity of a quantum channel is equal to the NS-assisted quantum capacity. It is worth noting that our result is equivalent to their result due to superdense coding [62] and teleportation [63].

Corollary 7 *For any quantum channel $\mathcal{N}(A \to B)$,*

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C_{\mathrm{NS}}^{(1)}(\mathcal{N}^{\otimes n},\epsilon) \le \max_{\rho_A} I(\rho_A;\mathcal{N}),$$

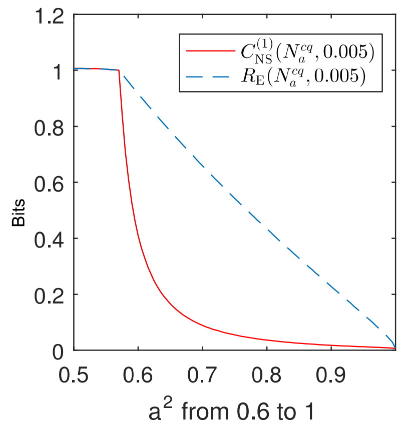

FIG. 4: When $\epsilon = 0.005$, $C_{\rm NS}^{(1)}(\mathcal{N}_a^{cq}, \epsilon)$ (red solid line) can be strictly smaller than the previous SDP bound $R_{\rm E}(\mathcal{N}_a^{cq}, \epsilon)$ (blue dashed line). Note that $C_{\rm NS\cap PPT}^{(1)}(\mathcal{N}_a^{cq}, \epsilon) = C_{\rm NS}^{(1)}(\mathcal{N}_a^{cq}, \epsilon)$ and $R_{\rm E\cap PPT}(\mathcal{N}_a^{cq}, \epsilon) = R_{\rm E}(\mathcal{N}_a^{cq}, \epsilon)$ in this case.

where $I(\rho_A; \mathcal{N}) := H(\rho_A) + H(\mathcal{N}(\rho_A)) - H((\mathrm{id} \otimes \mathcal{N})\phi_{\rho_A})$, *and* ϕ_{ρ_A} *is a purification of* ρ_A. *As a consequence,*

$$C_{\rm NS}(\mathcal{N}) = C_{\rm E}(\mathcal{N}).$$

Proof In [23], Matthews and Wehner prove that

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} R_{\rm E}(\mathcal{N}^{\otimes n}, \epsilon) \le \max_{\rho_A} I(\rho_A; \mathcal{N}).$$

By Proposition 6, we immediately obtain that

$$\begin{aligned} \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} C_{\rm NS}^{(1)}(\mathcal{N}^{\otimes n}, \epsilon) &\le \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} R_{\rm E}(\mathcal{N}^{\otimes n}, \epsilon) \\ &\le \max_{\rho_A} I(\rho_A; \mathcal{N}), \end{aligned} \tag{34}$$

which means that $C_{\rm NS}(\mathcal{N}) \le C_{\rm E}(\mathcal{N})$. Noticing that no-signalling codes are potentially stronger than the entanglement codes, it holds that $C_{\rm NS}(\mathcal{N}) \ge C_{\rm E}(\mathcal{N})$. Therefore, we have that $C_{\rm NS}(\mathcal{N}) = C_{\rm E}(\mathcal{N})$. □

C. Reduction to Polyanskiy-Poor-Verdú converse bound

For classical-quantum channels, the one-shot ϵ-error NS-assisted (or NS∩PPT-assisted) capacity can be further simplified based on the structure of the channel.

Proposition 8 *For the classical-quantum channel that acts as* $\mathcal{N} : x \to \rho_x$, *the Choi matrix of* $\mathcal{N}$ *is given by* $J_{\mathcal{N}} = \sum_x |x\rangle\langle x| \otimes \rho_x$. *Then, the SDP (25) of* $C_{\rm NS}^{(1)}(\mathcal{N}, \epsilon)$ *and the SDP (24) of* $C_{\rm NS\cap PPT}^{(1)}(\mathcal{N}, \epsilon)$ *can be* *simplified to*

$$\begin{aligned} C^{(1)}_{\text{NS}}(\mathcal{N},\epsilon) = C^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\epsilon) \\ = \log \max \sum s_x \\ \text{s.t. } 0 \le Q_x \le s_x \mathbb{1}_B, \forall x, \\ \sum_x Q_x = \mathbb{1}_B, \\ \sum_x \operatorname{Tr} Q_x \rho_x \ge \sum_x (1-\epsilon) s_x. \end{aligned} \tag{35}$$

Proof When $J_{\mathcal{N}} = \sum_x |x\rangle\langle x| \otimes \rho_x$, the SDP (25) easily simplifies to

$$\begin{aligned} C^{(1)}_{\text{NS}}(\mathcal{N},\epsilon) = -\log \min \eta \\ \text{s.t. } 0 \le F_x \le p_x \mathbb{1}_B, \forall x, \\ \sum_x p_x = 1, \\ \sum_x F_x/\eta = \mathbb{1}_B, \\ \sum_x \operatorname{Tr} F_x \rho_x \ge (1-\epsilon). \end{aligned} \tag{36}$$

By assuming that $Q_x = F_x/\eta$ and $s_x = p_x/\eta$, the above SDP simplifies to

$$\begin{aligned} C^{(1)}_{\text{NS}}(\mathcal{N},\epsilon) = \log \max \sum s_x \\ \text{s.t. } 0 \le Q_x \le s_x \mathbb{1}_B, \forall x, \\ \sum_x Q_x = \mathbb{1}_B, \\ \sum_x \operatorname{Tr} Q_x \rho_x \ge (1-\epsilon) \sum s_x, \end{aligned} \tag{37}$$

where we use the fact $\sum s_x = \sum p_x/\eta = 1/\eta$. One can use a similar method to simplify $C^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\epsilon)$ as well. □

Furthermore, for the classical channels, Polyanskiy, Poor, and Verdú [20] derive the finite block-length converse via hypothesis testing. In [22], an alternative proof of PPV converse was provided by considering the assistance of the classical no-signalling correlations. Here, we are going to show that both $C^{(1)}_{\text{NS}}(\mathcal{N},\epsilon)$ and $C^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\epsilon)$ will reduce to the PPV converse.

Let us first recall the linear program for the PPV converse bound of a classical channel $\mathcal{N}(y|x)$ [20, 22]:

$$\begin{aligned} R^{PPV}(\mathcal{N},\epsilon) = \max \sum_x s_x \\ \text{s.t. } Q_{xy} \le s_x, \forall x, y, \\ \sum_x Q_{xy} \le 1, \forall y, \\ \sum_{x,y} \mathcal{N}(y|x) Q_{xy} \ge (1-\epsilon) \sum_x s_x. \end{aligned} \tag{38}$$

For classical channels, we can further simplify the SDP (35) to a linear program which coincides with the Polyanskiy-Poor-Verdú converse bound.

Proposition 9 *For a classical channel* $\mathcal{N}(y|x)$*,*

$$C^{(1)}_{\text{NS}}(\mathcal{N},\epsilon) = C^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\epsilon) = R^{PPV}(\mathcal{N},\epsilon). \tag{39}$$

Proof The idea is to further simplify the SDP (35) via the structure of classical channels. For input x, the corresponding outputs can be seemed as $\rho_x = \sum_y \mathcal{N}(y|x)|y\rangle\langle y|$. Then, Q_x should be diagonal for any x, i.e., $Q_x = \sum_y Q_{xy}$. Thus, SDP (35) can be easily simplified to

$$\begin{aligned} C^{(1)}_{\text{NS}}(\mathcal{N},\epsilon) &= C^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\epsilon) \\ &= \log \max \sum_x s_x \\ &\quad \text{s.t. } Q_{xy} \le s_x, \forall x,y, \\ &\quad \sum_x Q_{xy} = 1, \forall y, \\ &\quad \sum_{x,y} \mathcal{N}(y|x) Q_{xy} \ge (1-\epsilon)\sum_x s_x. \end{aligned} \tag{40}$$

Using the similar technique in [22], the constraint $\sum_x Q_{xy} = 1$ can be relaxed to $\sum_x Q_{xy} \le 1$ in this case, which means that the linear program (40) is equal to the linear program (38). □

V. STRONG CONVERSE BOUNDS FOR CLASSICAL COMMUNICATION

A. SDP strong converse bounds for the classical capacity

It is well known that evaluating the classical capacity of a general channel is extremely difficult. To the best of our knowledge, the only known nontrivial strong converse bound for the classical capacity is the entanglement-assisted capacity [44] and there is also computable single-shot upper bound derived from entanglement measures [45]. In this section, we will derive two SDP strong converse bounds for the classical capacity of a general quantum channel. Our bounds are efficiently computable and do not depend on any special properties of the channel. We also show that for some classes of quantum channels, our bound can be strictly smaller than the entanglement-assisted capacity and the previous bound in [45].

Before introducing the strong converse bounds, we first show a single-shot SDP to estimate the optimal success probability of classical communication via multiple uses of the channel.

Proposition 10 *For any quantum channel $\mathcal{N}$ and given m,*

$$f_{\text{NS}\cap\text{PPT}}(\mathcal{N},m) \le f^+(\mathcal{N},m),$$

where

$$\begin{aligned} f^+(\mathcal{N},m) &= \min \ \operatorname{Tr} Z_B \\ &\quad \text{s.t. } -R_{AB} \le J_{\mathcal{N}}^{T_B} \le R_{AB}, \\ &\quad -m\mathbb{1}_A \otimes Z_B \le R_{AB}^{T_B} \le m\mathbb{1}_A \otimes Z_B. \end{aligned} \tag{41}$$

Furthermore, it holds that $f_{\text{NS}\cap\text{PPT}}(\mathcal{N}_1 \otimes \mathcal{N}_2, m_1 m_2) \le f^+(\mathcal{N}_1,m_1) f^+(\mathcal{N}_2,m_2)$. Consequently,

$$f_{\text{NS}\cap\text{PPT}}(\mathcal{N}^{\otimes n}, m^n) \le f^+(\mathcal{N},m)^n. \tag{42}$$

Proof We utilize the duality theory of semidefinite programming in the proof. To be specific, the dual SDP of $f^+(\mathcal{N},m)$ is given by

$$\begin{aligned} f^+(\mathcal{N},m) &= \max \ \operatorname{Tr} J_{\mathcal{N}}(V_{AB} - X_{AB})^{T_B} \\ &\quad \text{s.t. } V_{AB} + X_{AB} \le (W_{AB} - Y_{AB})^{T_B}, \\ &\quad \operatorname{Tr}_A(W_{AB} + Y_{AB}) \le \mathbb{1}_B/m, \\ &\quad V_{AB}, X_{AB}, W_{AB}, Y_{AB} \ge 0. \end{aligned} \tag{43}$$

It is worth noting that the optimal values of the primal and the dual SDPs above coincide. This is a consequence of strong duality. By Slater's condition, one simply needs to show that there exists positive definite V_{AB}, X_{AB}, W_{AB} and Y_{AB} such that $V_{AB} + X_{AB} < (W_{AB} - Y_{AB})^{T_B}$ and $\operatorname{Tr}_A(W_{AB} + Y_{AB}) < \mathbb{1}_B/m$, which holds for $W_{AB} = 2Y_{AB} = 5V_{AB} = X_{AB} = \mathbb{1}_{AB}/2md_A$.

In SDP (43), let us choose $X_{AB} = Y_{AB} = 0$ and $V_{AB}^{T_B} = W_{AB}$, then we have that

$$f^+(\mathcal{N}, m) \geq K \geq f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}, m), \tag{44}$$

where $K := \max \operatorname{Tr} J_{\mathcal{N}} W_{AB} \mathrm{s.t.} \mathrm{W_{AB}}, \mathrm{W_{AB}^{T_B}} \geq 0, \mathrm{Tr_A}\, \mathrm{W_{AB}} \leq \mathbb{1}_{\mathrm{B}}/\mathrm{m}$. This means that the SDP (43) of $f^+(\mathcal{N}, m)$ is a relaxation of the SDP (8) of $f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}, m)$.

To see $f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}_1 \otimes \mathcal{N}_2, m_1 m_2) \leq f^+(\mathcal{N}_1, m_1) f^+(\mathcal{N}_2, m_2)$, we first suppose that the optimal solution to SDP (41) of $f^+(\mathcal{N}_1, m_1)$ is $\{Z_1, R_1\}$ and the optimal solution to SDP (41) of $f^+(\mathcal{N}_2, m_2)$ is $\{Z_2, R_2\}$. Let us denote the Choi-Jamiołkowski matrix of $\mathcal{N}_1$ and $\mathcal{N}_2$ by J_1 and J_2, respectively. It is easy to see that $R_1 \otimes R_2 \pm J_1^{T_B} \otimes J_2^{T_{B'}} \geq 0$ since

$$\begin{aligned}
&R_1 \otimes R_2 + J_1^{T_B} \otimes J_2^{T_{B'}} \\
=&\frac{1}{2}[(R_1 + J_1^{T_B}) \otimes (R_2 + J_2^{T_{B'}}) + (R_1 - J_1^{T_B}) \otimes (R_2 - J_2^{T_{B'}})], \\
&R_1 \otimes R_2 - J_1^{T_B} \otimes J_2^{T_{B'}} \\
=&\frac{1}{2}[(R_1 + J_1^{T_B}) \otimes (R_2 - J_2^{T_{B'}}) + (R_1 - J_1^{T_B}) \otimes (R_2 + J_2^{T_{B'}})].
\end{aligned}$$

Therefore, we have that

$$-R_1 \otimes R_2 \leq J_1^{T_B} \otimes J_2^{T_{B'}} \leq R_1 \otimes R_2.$$

Applying similar techniques, it is easy to prove that

$$-m_1 m_2 \mathbb{1}_{AA'} \otimes Z_1 \otimes Z_2 \leq R_1^{T_B} \otimes R_2^{T_{B'}} \leq m_1 m_2 \mathbb{1}_{AA'} \otimes Z_1 \otimes Z_2.$$

Hence, $\{Z_1 \otimes Z_2, R_1 \otimes R_2\}$ is a feasible solution to the SDP (41) of $f^+(\mathcal{N}_1\mathcal{N}_2, m_1 m_2)$, which means that

$$\begin{aligned}
f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}_1\mathcal{N}_2, m_1 m_2) &\leq f^+(\mathcal{N}_1\mathcal{N}_2, m_1 m_2) \\
&\leq \operatorname{Tr} Z_1 \otimes Z_2 = f^+(\mathcal{N}_1, m_1) f^+(\mathcal{N}_2, m_2).
\end{aligned}$$

□

Now, we are able to derive the strong converse bounds of the classical capacity.

Theorem 11 *For any quantum channel $\mathcal{N}$,*

$$\begin{aligned}
C(\mathcal{N}) \leq& C_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}) \\
\leq& C_\beta(\mathcal{N}) \log \beta(\mathcal{N}) \leq \log(d_B \|J_{\mathcal{N}}^{T_B}\|_\infty),
\end{aligned}$$

where

$$\begin{aligned}
\beta(\mathcal{N}) = &\min \operatorname{Tr} S_B \\
&\text{s.t.} - R_{AB} \leq J_{\mathcal{N}}^{T_B} \leq R_{AB}, \\
&\quad - \mathbb{1}_A \otimes S_B \leq R_{AB}^{T_B} \leq \mathbb{1}_A \otimes S_B.
\end{aligned} \tag{45}$$

In particular, when the communication rate exceeds $C_\beta(\mathcal{N})$, the error probability goes to one exponentially fast as the number of channel uses increases.

Proof For n uses of the channel, we suppose that the rate of the communication is r. By Proposition 10, we have that

$$f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}^{\otimes n}, 2^{rn}) \le f^{+}(\mathcal{N}, 2^{r})^{n}. \tag{46}$$

Therefore, the n-shot error probability satisfies that

$$\epsilon_n = 1 - f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}^{\otimes n}, 2^{rn}) \ge 1 - f^{+}(\mathcal{N}, 2^{r})^{n}. \tag{47}$$

Suppose that the optimal solution to the SDP (45) of $\beta(\mathcal{N})$ is $\{S_0, R_0\}$. It is easy to verify that $\{S_0/\operatorname{Tr} S_0, R_0\}$ is a feasible solution to the SDP (41) of $f^{+}(\mathcal{N}, \operatorname{Tr} S_0)$. Therefore,

$$f^{+}(\mathcal{N}, \beta(\mathcal{N})) \le \operatorname{Tr}(S_0/\operatorname{Tr} S_0) = 1.$$

It is not difficult to see that $f^{+}(\mathcal{N}, m)$ monotonically decreases when m increases. Thus, for any $2^r > \beta(\mathcal{N})$, we have $f^{+}(\mathcal{N}, 2^r) < 1$. Then, by Eq. (47), it is clear that the corresponding n-shot error probability ϵ_n will go to one exponentially fast as n increases. Hence, $C_\beta(\mathcal{N})$ is a strong converse bound for the NS∩PPT-assisted classical capacity of $\mathcal{N}$.

Furthermore, let us choose $R_{AB} = \|J_{\mathcal{N}}^{T_B}\|_\infty \mathbb{1}_{AB}$ and $S_B = \|J_{\mathcal{N}}^{T_B}\|_\infty \mathbb{1}_B$. It is clear that $\{R_{AB}, S_B\}$ is a feasible solution to the SDP (45) of $\beta(\mathcal{N})$, which means that $\beta(\mathcal{N}) \le d_B \|J_{\mathcal{N}}^{T_B}\|_\infty$. □

Remark C_β has some remarkable properties. For example, it is additive: $C_\beta(\mathcal{N}_1 \otimes \mathcal{N}_2) = C_\beta(\mathcal{N}_1) + C_\beta(\mathcal{N}_2)$ for different quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$. This can be proved by utilizing semidefinite programming duality.

With similar techniques, we are going to show another SDP strong converse bound for the classical capacity of a general quantum channel.

Theorem 12 *For a quantum channel $\mathcal{N}$, we derive the following strong converse bound for the NS∩PPT assisted classical capacity, i.e.,*

$$C(\mathcal{N}) \le C_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}) \le C_\zeta(\mathcal{N}) = \log \zeta(\mathcal{N})$$

with

$$\begin{aligned} \zeta(\mathcal{N}) = & \min \operatorname{Tr} S_B \\ & \text{s.t.} V_{AB} \ge J_{\mathcal{N}}, -\mathbb{1}_A \otimes S_B \le V_{AB}^{T_B} \le \mathbb{1}_A \otimes S_B \end{aligned} \tag{48}$$

And if the communication rate exceeds $C_\zeta(\mathcal{N})$, the error probability will go to one exponentially fast as the number of channel uses increase.

Proof We first introduce the following SDP to estimate the optimal success probability:

$$\begin{aligned} \widetilde{f}^{+}(\mathcal{N}, m) = \min\ & \operatorname{Tr} S_B \\ \text{s.t. } & V_{AB} \ge J_{\mathcal{N}}, \\ & - m\mathbb{1}_A \otimes S_B \le V_{AB}^{T_B} \le m\mathbb{1}_A \otimes S_B. \end{aligned} \tag{49}$$

Similar to Proposition 10, we can prove that

$$f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}^{\otimes n}, m^n) \le \widetilde{f}^{+}(\mathcal{N}, m)^n. \tag{50}$$

Then, when the communication rate exceeds $C_\zeta(\mathcal{N})$, we can use the technique in Theorem 11 to prove that the error probability will go to one exponentially fast as the number of channel uses increase. □

As an example, we first apply our bounds to the qudit noiseless channel. In this case, the bounds are tight and strictly smaller than the entanglement-assisted classical capacity.

Proposition 13 *For the qudit noiseless channel $I_d(\rho)=\rho$, it holds that*

$$C(I_d)=C_\beta(I_d)=C_\zeta(I_d)=\log d<2\log d=C_{\rm E}(I_d). \tag{51}$$

Proof It is clear that $C(I_d)\geq \log d$. By the fact that $\|J_{I_d}^{T_B}\|_\infty=1$, it is easy to see that $C_\beta(I_d)\leq \log d\|J_{I_d}^{T_B}\|_\infty=\log d$. Similarly, we also have $C_\zeta(I_d)\leq\log d$. And $C_{\rm E}(I_d)=2\log d$ is due to the superdense coding [62]. □

B. Amplitude damping channel

For the amplitude damping channel $\mathcal{N}_\gamma^{AD}=\sum_{i=0}^1 E_i\cdot E_i^\dagger$ $(0\leq\gamma\leq1)$ with $E_0=|0\rangle\langle0|+\sqrt{1-\gamma}|1\rangle\langle1|$ and $E_1=\sqrt{\gamma}|0\rangle\langle1|$, the Holevo capacity $\chi(\mathcal{N}_\gamma^{AD})$ is given in [64]. However, its classical capacity remains unknown. The only known nontrivial and meaningful upper bound for the classical capacity of the amplitude damping channel was established in [45]. As an application of theorems 11 and 12, we show a strong converse bound for the classical capacity of the qubit amplitude damping channel. Remarkably, our bound improves the best previously known upper bound [45].

Theorem 14 *For amplitude damping channel $\mathcal{N}_\gamma^{AD}$,*

$$C_{\rm NS\cap PPT}(\mathcal{N}_\gamma^{AD})\leq C_\zeta(\mathcal{N}_\gamma^{AD})=C_\beta(\mathcal{N}_\gamma^{AD})=\log(1+\sqrt{1-\gamma}).$$

As a consequence,

$$C(\mathcal{N}_\gamma^{AD})\leq\log(1+\sqrt{1-\gamma}).$$

Proof Suppose that

$$S_B=\frac{\sqrt{1-\gamma}+1+\gamma}{2}|0\rangle\langle0|+\frac{\sqrt{1-\gamma}+1-\gamma}{2}|1\rangle\langle1|$$

and

$$V_{AB}=J_\gamma^{AD}+(\sqrt{1-\gamma}-1+\gamma)|v\rangle\langle v|$$

with $|v\rangle=\frac{1}{\sqrt2}(|00\rangle+|11\rangle)$.

It is clear that $V_{AB}\geq J_\gamma^{AD}$. Moreover, it is easy to see that

$$\mathbb{1}_A\otimes S_B-V_{AB}^{T_B}=\frac{\sqrt{1-\gamma}+1-\gamma}{2}(|01\rangle-|10\rangle)(\langle01|-\langle10|)\geq0$$

and $\mathbb{1}_A\otimes S_B+V_{AB}^{T_B}=(\sqrt{1-\gamma}+1+\gamma)|00\rangle\langle00|+(\sqrt{1-\gamma}+1-\gamma)|11\rangle\langle11|+\frac{\sqrt{1-\gamma}+1-\gamma}{2}(|01\rangle\langle01|+|01\rangle\langle10|+|10\rangle\langle01|)+\frac{\sqrt{1-\gamma}+1+3\gamma}{2}|10\rangle\langle10|\geq0$.

Therefore, $\{S_B,V_{AB}\}$ is a feasible solution to SDP (48), which means that

$$C_\zeta(\mathcal{N}_\gamma^{AD})\leq\log\operatorname{Tr}S_B=\log(1+\sqrt{1-\gamma}).$$

One can also use the dual SDP of C_β to show that $C_\beta(\mathcal{N}_\gamma^{AD})\geq\log(1+\sqrt{1-\gamma})$. Hence, we have that $C_\zeta(\mathcal{N}_\gamma^{AD})=\log(1+\sqrt{1-\gamma})$.

Similarly, it can also be calculated that $C_\beta(\mathcal{N}_\gamma^{AD})=\log(1+\sqrt{1-\gamma})$. □

Remark: It is worth noting that our bound is strictly smaller than the entanglement-assisted capacity when $\gamma\leq0.75$ as shown in the following FIG. 5. We further compare our bound with the previous upper bound [45] and lower bound [64] in FIG. 6. The authors of [64] show that $C(\mathcal{N}_\gamma^{AD})\geq\max_{0\leq p\leq1}\{H_2[(1-\gamma)p]-H_2[(1+\sqrt{1-4(1-\gamma)\gamma p^2})/2]\}$, where H_2 is the binary entropy. It is clear that our bound provides a tighter bound to the classical capacity than the previous bound [45].

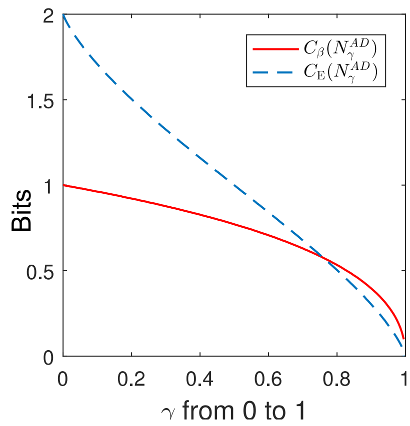

FIG. 5: The solid line depicts $C_\beta(\mathcal{N}_\gamma^{AD})$ while the dashed line depicts $C_{\mathrm{E}}(\mathcal{N}_\gamma^{AD})$. It is worth noting that $C_\beta(\mathcal{N}_\gamma^{AD})$ is strictly smaller than $C_{\mathrm{E}}(\mathcal{N}_\gamma^{AD})$ for any $\gamma \leq 0.75$.

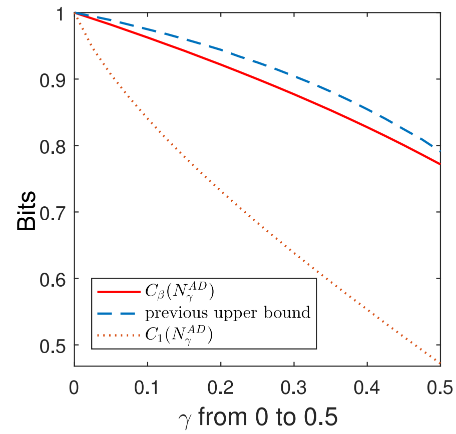

FIG. 6: The solid line depicts $C_\beta(\mathcal{N}_\gamma^{AD})$, the dashed line depicts the previous bound of $C(\mathcal{N}_\gamma^{AD})$ [45], and the dotted line depicts the lower bound [64]. Our bound is tighter than the previous bound in [45].

C. Strong converse property for a new class of quantum channels

In [65], a class of qutrit-to-qutrit channels was introduced to show the separation between quantum Lovász number and entanglement-assisted zero-error classical capacity. It turns out that this class of channels also has strong converse property for classical or private communication. To be specific, the channel from register A to B is given by $\mathcal{N}_\alpha(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$ $(0 < \alpha \leq \pi/4)$ with

$$E_0 = \sin\alpha|0\rangle\langle 1| + |1\rangle\langle 2|, E_1 = \cos\alpha|2\rangle\langle 1| + |1\rangle\langle 0|.$$

It follows that the complementary channel of $\mathcal{N}_\alpha$ is $\mathcal{N}_\alpha^c(\rho) = \sum_{i=0}^2 F_i\rho F_i^\dagger$ with

$$F_0 = \sin\alpha|0\rangle\langle 1|, F_1 = |0\rangle\langle 2| + |1\rangle\langle 0|, F_2 = \cos\alpha|1\rangle\langle 1|.$$

Proposition 15 *For $\mathcal{N}_\alpha$ $(0 < \alpha \leq \pi/4)$, we have that*

$$C(\mathcal{N}_\alpha) = C_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}_\alpha) = C_\beta(\mathcal{N}_\alpha) = 1.$$

Proof Suppose the $Z_B = \sin^2\alpha|0\rangle\langle 0| + \cos^2\alpha|2\rangle\langle 2| + |1\rangle\langle 1|$ and

$$\begin{aligned} R_{AB} = &|01\rangle\langle 01| + |11\rangle\langle 11| + |21\rangle\langle 21| + \sin^2\alpha(|10\rangle\langle 10| + |20\rangle\langle 20|) \\ &+ \cos^2\alpha(|02\rangle\langle 02| + |12\rangle\langle 12|) + \sin\alpha\cos\alpha(|02\rangle\langle 20| + |20\rangle\langle 02|). \end{aligned}$$

It is easy to check that

$$-R_{AB} \leq J_{\mathcal{N}_\alpha}^{T_B} \leq R_{AB} \text{ and } -\mathbb{1}_A \otimes Z_B \leq R_{AB}^{T_B} \leq \mathbb{1}_A \otimes Z_B,$$

where $J_{\mathcal{N}_\alpha}$ is the Choi-Jamiołkowski matrix of $\mathcal{N}_\alpha$.

Therefore, $\{Z_B, R_{AB}\}$ is a feasible solution of SDP (45) of $\beta(\mathcal{N}_\alpha)$, which means that

$$\beta(\mathcal{N}_\alpha) \leq \operatorname{Tr} Z_B = 2.$$

Noticing that we can use input $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ to transmit two messages via $\mathcal{N}$, we can conclude that

$$C(\mathcal{N}_\alpha) = C_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}_\alpha) = 1.$$

□

Remark In [65], the entanglement-assisted capacity of $\mathcal{N}_\alpha$ is shown to be

$$C_{\mathrm{E}}(\mathcal{N}_\alpha) = 2.$$

Therefore, for $\mathcal{N}_\alpha$ $(0 < \alpha \leq \pi/4)$, our bound C_β is strictly smaller than the entanglement-assisted capacity. In this case, we also note that $C_\beta(\mathcal{N}_\alpha) < C_\zeta(\mathcal{N}_\alpha)$. However, it remains unknown whether C_β is always smaller than or equal to C_ζ.

Furthermore, it is easy to see that $\mathcal{N}_\alpha$ is neither an entanglement-breaking channel nor a Hadamard channel. Note also that $\mathcal{N}_\alpha$ is not belong to the three classes in [17], for which the strong converse for classical capacity has been established. Thus, our results show a new class of quantum channels which satisfy the strong converse property for classical capacity. □

Moreover, we find that the strong converse property also holds for the private classical capacity [66, 67] of $\mathcal{N}_\alpha$. Note that private capacity requires that no information leaked to the environment and is usually called $P(\mathcal{N})$. Recently, several converse bounds for private communication were established in [68–72].

Proposition 16 *The private capacity of $\mathcal{N}_\alpha$ is exactly one bit, i.e., $P(\mathcal{N}_\alpha) = 1$. In particular,*

$$Q(\mathcal{N}_\alpha) \leq \log(1+\cos\alpha) < 1 = P(\mathcal{N}_\alpha) = C(\mathcal{N}_\alpha) = \frac{1}{2}C_E(\mathcal{N}_\alpha).$$

Proof On one hand, it is easy to see that $P(\mathcal{N}_\alpha) \leq C(\mathcal{N}_\alpha) = C_\beta(\mathcal{N}_\alpha) = 1$.

On the other hand, Alice can choose two input states $|\psi_0\rangle = |1\rangle$ and $|\psi_1\rangle = \cos\alpha|0\rangle + \sin\alpha|2\rangle$, then the corresponding output states Bob received are

$$\mathcal{N}_\alpha(|\psi_0\rangle\langle\psi_0|) = \sin\alpha^2|0\rangle\langle 0| + \cos\alpha^2|2\rangle\langle 2|,$$
$$\mathcal{N}_\alpha(|\psi_1\rangle\langle\psi_1|) = |1\rangle\langle 1|.$$

It is clear that Bob can perfectly distinguish these two output states. Meanwhile, the corresponding outputs of the complementary channel $\mathcal{N}_\alpha^c$ are same, i.e.,

$$\mathcal{N}_\alpha^c(|\psi_0\rangle\langle\psi_0|) = \mathcal{N}_\alpha^c(|\psi_1\rangle\langle\psi_1|) = \sin\alpha^2|0\rangle\langle 0| + \cos\alpha^2|1\rangle\langle 1|,$$

which means that the environment obtain zero information during the communication.

Applying the SDP bound of the quantum capacity in [73], the quantum capacity of $\mathcal{N}_\alpha$ is strictly smaller than $\log(1+\cos\alpha)$. □

Our result establishes the strong converse property for both the classical and private capacities of $\mathcal{N}_\alpha$. For the classical capacity, such a property was previously only known for classical channels, identity channel, entanglement-breaking channels, Hadamard channels and particular covariant quantum channels [17, 19]. For the private capacity, such a property was previously only known for generalized dephasing channels and quantum erasure channels [68]. Moreover, our result also shows a simple example of the distinction between the private and the quantum capacities, which were discussed in [74, 75].

VI. ZERO-ERROR CAPACITY

While ordinary information theory focuses on sending messages with asymptotically vanishing errors [76], Shannon also investigated this problem in the zero-error setting and described the zero-error capacity of a channel as the maximum rate at which it can be used to transmit information with zero probability of error [46]. Recently the zero-error information theory has been studied in the quantum setting and many new interesting phenomena have been found [47, 77–83].

The one-shot zero-error capacity of a quantum channel $\mathcal{N}$ is the maximum number of inputs such that the receiver can perfectly distinguish the corresponding output states. Cubitt et al. [84] first introduced the zero-error communication via classical channels assisted by classical no-signalling correlations. Recently, no-signalling-assisted zero-error communication over quantum channels was introduced in [42].

Using the expression (24) for our one-shot ϵ-error capacity, we are going to show a formula for the one-shot zero-error classical capacity assisted by NS (or NS∩PPT) codes.

Theorem 17 *The one-shot zero-error classical capacity (quantified as messages) of $\mathcal{N}$ assisted by* $\mathrm{NS}\cap\mathrm{PPT}$ *codes is given by*

$$\begin{aligned} M_{0,\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}) = \max\ & \operatorname{Tr} S_A \\ \text{s.t. } & 0 \le U_{AB} \le S_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr}_A U_{AB} = \mathbb{1}_B, \\ & \operatorname{Tr} J_{\mathcal{N}}(S_A \otimes \mathbb{1}_B - U_{AB}) = 0, \\ & 0 \le U_{AB}^{T_B} \le S_A \otimes \mathbb{1}_B \ (\mathrm{PPT}). \end{aligned} \tag{52}$$

To obtain $M_{0,\mathrm{NS}}(\mathcal{N})$, one only needs to remove the PPT constraint. By the regularization, the Ω-assisted zero-error classical capacity is

$$C_{0,\Omega}(\mathcal{N}) = \sup_{n\ge1} \frac{1}{n} \log M_{0,\Omega}(\mathcal{N}^{\otimes n}).$$

Proof When $\epsilon = 0$, it is easy to see that

$$\begin{aligned} C^{(1)}_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}, 0) = -\log\min\ & \eta \\ \text{s.t. } & 0 \le F_{AB} \le \rho_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr} \rho_A = 1, \operatorname{Tr}_A F_{AB} = \eta \mathbb{1}_B, \\ & \operatorname{Tr} J_{\mathcal{N}} F_{AB} \ge 1, \\ & 0 \le F_{AB}^{T_B} \le \rho_A \otimes \mathbb{1}_B. \end{aligned} \tag{53}$$

Then, assuming that $x = 1/\eta$, $U_{AB} = xF_{AB}$ and $S_A = x\rho_A$, we have that

$$\begin{aligned} M_{0,\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}) &= 2^{C^{(1)}_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N},0)} \\ &= \max\ \operatorname{Tr} S_A \\ &\quad \text{s.t. } 0 \le \mathrm{U_{AB}} \le \mathrm{S_A} \otimes \mathbb{1}_\mathrm{B}, \\ &\quad \operatorname{Tr}_A U_{AB} = \mathbb{1}_B, \\ &\quad \operatorname{Tr} J_{\mathcal{N}} U_{AB} \ge \operatorname{Tr} S_A, \\ &\quad 0 \le U_{AB}^{T_B} \le S_A \otimes \mathbb{1}_B. \end{aligned} \tag{54}$$

By the fact that $\operatorname{Tr} S_A = \operatorname{Tr} J_{\mathcal{N}}(S_A \otimes \mathbb{1}_B)$, the third constraint in Eq. (54) is equivalent to $\operatorname{Tr} J_{\mathcal{N}}(S_A \otimes \mathbb{1}_B - U_{AB}) \le 0$. Noticing that $S_A \otimes \mathbb{1}_B - U_{AB} \ge 0$, we can simplify Eq. (54) to

$$\begin{aligned} M_{0,\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}) = \max\ & \operatorname{Tr} S_A \\ \text{s.t. } & 0 \le U_{AB} \le S_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr}_A U_{AB} = \mathbb{1}_B, \\ & \operatorname{Tr} J_{\mathcal{N}}(S_A \otimes \mathbb{1}_B - U_{AB}) = 0, \\ & 0 \le U_{AB}^{T_B} \le S_A \otimes \mathbb{1}_B. \end{aligned} \tag{55}$$

□

Remark It is worth noting that $M_{0,\mathrm{NS}}(\mathcal{N})$ coincides with the no-signalling assisted zero-error capacity in [42]. Also, it can be proved that $M_{0,\mathrm{NS}\cap\mathrm{PPT}}$ also depends only on the non-commutative bipartite graph [42] of $\mathcal{N}$. □

A natural application of $M_{0,\Omega}(\mathcal{N})$ is to upper bound the one-shot zero-error capacity, i.e.,

$$M_0(\mathcal{N}) \le M_{0,\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}) \le M_{0,\mathrm{NS}}(\mathcal{N}).$$

It is known that computing the one-shot zero-error capacity of a quantum channel is QMA-complete [13]. However, our bounds can be efficiently solved by semidefinite programming. To the best of our knowledge, for a general quantum channel $\mathcal{N} = \sum_i E_i \cdot E_i^\dagger$, the best known bound of the one-shot zero-error capacity is the naive form of the Lovász number $\vartheta(\mathcal{N})$ in [47], i.e.,

$$\vartheta(\mathcal{N}) = \vartheta(\mathcal{S}) = \max\{\|\mathbb{1} + T\|_\infty : T \in \mathcal{S}^\perp, \mathbb{1} + T \ge 0\},$$

where $\mathcal{S} = \operatorname{span}\{E_j^\dagger E_k\}$ is the non-commutative graph of $\mathcal{N}$.

In the next Proposition, we show that $M_{0,\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N})$ can be strictly smaller than $\vartheta(\mathcal{N})$ for some quantum channel $\mathcal{N}$. This implies that $M_{0,\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N})$ can provide a more accurate estimation of the one-shot zero-error capacity of some general quantum channels.

Proposition 18 *For* $\mathcal{N}_\alpha$ $(0 < \alpha \le \pi/4)$,

$$M_{0,\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}_\alpha) < \vartheta(\mathcal{N}_\alpha).$$

Proof One one hand, one can also use the prime and dual SDPs of $M_{0,\mathrm{NS}\cap\mathrm{PPT}}$ to prove $M_{0,\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}_\alpha) \le 2$. Indeed, this is also easy to see by Proposition 15.

On the other hand, we are going to prove $\vartheta(\mathcal{N}_\alpha) \ge 1 + \cos^{-2}\alpha$. Suppose that $T_0 = -|0\rangle\langle 0| + \cos^{-2}\alpha|1\rangle\langle 1| + (1 - \cos^{-2}\alpha)|2\rangle\langle 2|$. It is clear that $T_0 \in S^\perp$ and $\mathbb{1} + T_0 \ge 0$. Thus,

$$\vartheta(\mathcal{N}_\alpha) \ge \|\mathbb{1} + T_0\|_\infty = \langle 1|(\mathbb{1} + T_0)|1\rangle = 1 + \cos^{-2}\alpha > 2.$$

□

For this class of quantum channels, it is worth noting that the private zero-error capacity is also one bit while its quantum zero-error capacity is strictly smaller than one qubit, i.e., $Q_0(\mathcal{N}_\alpha) < 1 = P_0(\mathcal{N}_\alpha) = C_0(\mathcal{N}_\alpha)$. This shows a difference between the quantum and the private capacities of a quantum channel in the zero-error setting, which relates to the work about maximum privacy without coherence in the zero-error case [85].

VII. CONCLUSIONS AND DISCUSSIONS

In summary, we have established fundamental limits for classical communication over quantum channels by considering general codes with NS constraint or NS∩PPT constraint. New SDP bounds for classical communication under both finite blocklength and asymptotic settings are obtained in this work.

We first study the finite blocklength regime. By imposing both no-signalling and PPT-preserving constraints, we have obtained the optimal success probabilities of transmitting classical information assisted by NS and NS∩PPT codes. Based on this, we have also derived the one-shot ϵ-error NS-assisted and NS∩PPT-assisted capacities. In particular, all of these one-shot characterizations are in the form of semidefinite programs. The one-shot NS-assisted and NS∩PPT-assisted) ϵ-error capacities provide an improved finite blocklength estimation of the classical communication than the previous quantum hypothesis testing converse bounds in [23]. Moreover, for classical channels, the one-shot NS-assisted and NS∩PPT-assisted ϵ-error capacities are equal to the linear program for the Polyanskiy-Poor-Verdú converse bound [20, 22], thus giving an alternative proof of that result. Furthermore, in the asymptotic regime, we derive two SDP strong converse bounds of the classical capacity of a general quantum channel, which are efficiently computable and can be strictly smaller than the entanglement-assisted capacity. As an example, we have shown an improved upper bound on the classical capacity of the qubit amplitude damping channel. Moreover, we have proved that the strong converse property holds for both classical and private capacities for a new class of quantum channels. This result may help us deepen the understanding of the limit ability of a quantum channel to transmit classical information.

Finally, we apply our results to the study of zero-error capacity. To be specific, based on our SDPs of optimal success probability, we have derived the one-shot NS-assisted (or NS∩PPT-assisted) zero-error capacity. Our result of NS-assisted capacity provides an alternative derivation for the NS-assisted zero-error capacity in [42]. Moreover, the one-shot NS∩PPT-assisted zero-error capacity also provide some insights in quantum zero-error information theory.

It would be interesting to study the asymptotic capacity $C_{\mathrm{NS\cap PPT}}$ using such techniques as quantum hypothesis testing. Maybe it also has a single-letter formula similar to entanglement-assisted classical capacity. Perhaps one can obtain tighter converse bounds via the study of $C_{\mathrm{NS\cap PPT}}$. Another direction is to further tighten the one-shot and strong converse bounds by involving the separable constraint [52]. It would also be interesting to study how to implement the no-signalling and PPT-preserving codes.

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