Quantum Channel Simulation and the Channel's Smooth Max-Information

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***Abstract*—We study the general framework of quantum channel simulation, that is, the ability of a quantum channel to simulate another one using different classes of codes. Our main results are as follows. First, we show that the minimum error of simulation under non-signalling assisted codes is efficiently computable via semidefinite programming. The cost of simulating a channel via noiseless quantum channels under non-signalling assisted codes can also be characterized as a semidefinite program. Second, we introduce the channel's smooth max-information, which can be seen as a one-shot generalization of the channel's mutual information. We show that the one-shot quantum simulation cost under non-signalling assisted codes is exactly equal to the channel's smooth max-information. Due to the quantum reverse Shannon theorem, the channel's smooth max-information converges to the channel's mutual information in the independent and identically distributed asymptotic limit. Together with earlier findings on the (activated) non-signalling assisted one-shot capacity of channels [Wang et al., arXiv:1709.05258], this suggest that the operational min- and max-type one-shot analogues of the channel's mutual information are the channel's hypothesis testing relative entropy and the channel's smooth max-information, respectively.**

I. Introduction

Channel simulation is a fundamental problem in information theory. It concerns how to use a channel $\mathcal{N}$ from Alice (A) to Bob (B) to simulate another channel $\mathcal{M}$ also from A to B [1]. Shannon's noisy channel coding theorem determines the capability of noisy classical channels to simulate noiseless ones [2]. Dual to this famous coding theorem, the 'reverse Shannon theorems' concerns the use of noiseless channels to simulate noisy ones as well as the use of a channel to simulate another [3]. Depending on the different resources available between A and B, this simulation problem has many variants. For classical channels, the Classical Reverse Shannon Theorem [3] states that every channel can be simulated using an amount of classical communication equal to the capacity of the channel when there is free shared randomness between A and B in the asymptotic setting. For quantum channels, the case when A and B share an unlimited amount of entanglement has been completely solved by the Quantum Reverse Shannon Theorem (QRST) [4], [5], which states that the rate to optimally simulate a quantum channel in the asymptotic setting is determined by its entanglement-assisted classical capacity. Moreover, as an analog to the classical scenario [4], [6], the QRST gives rise to a strong converse for the entanglement-assisted classical capacity of quantum channels. In the zero-error setting [7], using one channel to simulate another exactly with the aid of no-signalling correlations has been studied recently in Refs. [8]–[10]. Moreover, quantum channel simulations via entanglement and quantum coherence have been studied in Ref. [11] and Ref. [12], respectively.

However, in a realistic setting, the number of channel uses is necessarily limited, and it is difficult to satisfy the zero-error constraint. More importantly, it is not easy to perform encoding and decoding circuits coherently over large numbers of qubits in the near future. Therefore, it is important to characterize how well we can simulate a quantum channel from another with finite resources. The first step in this direction is to consider the one-shot setting. One-shot analysis has recently attracted great interest in classical and quantum information theory, see, e.g., [13]–[24].

In Sect. III, we study the framework for quantum channel simulation in the one-shot regime, where one has access only to a single use of the quantum channel. In particular, we characterize the minimum error of simulation under no-signalling (NS) and positive-partial-transpose-preserving (PPT) assisted codes as a semidefinite program (SDP) [25] which is efficiently computable. The cost of simulating a channel via noiseless quantum channels under NS-assisted codes can also be characterized as an SDP.

In the setting of the entanglement-assisted one-shot capacity of quantum channels, Matthews and Wehner gave a converse bound in terms of the channel's hypothesis testing relative entropy [18]. Moreover, a subset of us recently showed that the activated NS-assisted one-shot capacity is exactly given by the channels hypothesis testing relative entropy [22] – generalizing the corresponding classical results [13], [26]. As a counterpart, our second main result shows that the one-shot quantum simulation cost under NS-assisted codes is given by a novel entropic measure, which we call the channel's smooth max-information.

The smooth max-information of a state was introduced in [5] as a one-shot generalization of the mutual information [27]. It is a useful quantity in one-shot information theory, quantum rate distortion theory, and the physics of quantum many-body systems. However, there are multiple ways to define the smooth max-information of quantum states [28].

In Sect. IV, we introduce the channel's smooth max-information, which can be seen as a one-shot generalization of channel's mutual information. Notably, we show that the channel's smooth max-information has the operational interpretation as the one-shot quantum simulation cost under NS-assisted codes. Due to the QRST, the channel's smooth max-information converges to the channel's mutual information in the asymptotic limit of many independent and identically distributed copies. The channel's smooth max-information is also monotone under composition with completely positive and trace-preserving linear maps.

II. Channel simulation and codes

Let us now formally introduce the task of channel simulation and some notation. A quantum channel $\mathcal{N}_{A_o\to B_i}$ is a completely positive (CP) and trace-preserving (TP) linear map from operators on a finite-dimensional Hilbert space A_o to operators on a finite-dimensional Hilbert space B_i. As shown in Fig. 1, Alice and Bob share a quantum channel $\mathcal{N}_{A_o\to B_i}$. By adding encoding and decoding scheme, they can use channel $\mathcal{N}$ to simulate another channel $\mathcal{M}$. Composing with the encoder and decoder, their effective channel is given by $\widetilde{\mathcal{N}}_{A_i\to B_o}=\Pi_{A_iB_i\to A_oB_o}\circ\mathcal{N}_{A_o\to B_i}$, where Π is a bipartite quantum operation that generalizes the usual encoding scheme $\mathcal{E}$ and decoding scheme $\mathcal{D}$. Note that the bipartite quantum operation Π here is required to be B to A no-signalling, which makes the composition of Π and $\mathcal{N}$ feasible [9], [29]. We say such Π is an Ω-assisted code if it can be implemented by local operations with Ω-assistance. In the following, we eliminate Ω for the case of unassisted codes. We write $\Omega=\mathrm{NS}$ and $\Omega=\mathrm{PPT}$ for NS-assisted and PPT-assisted codes, respectively. These codes have also been applied to study other tasks of quantum information processing (e.g., [26], [30], [32]–[36]). In particular,

- an unassisted code reduces to the product of encoder and decoder, i.e., $\Pi=\mathcal{D}_{B_i\to B_o}\mathcal{E}_{A_i\to A_o}$
- a NS-assisted code corresponds to a bipartite operation which is no-signalling from Alice to Bob and vice-versa
- a PPT-assisted code corresponds to a bipartite operation whose Choi-Jamiołkowski matrix is positive under partial transpose over systems B_iB_o.

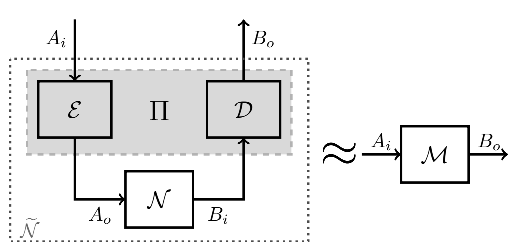

Fig. 1. General framework of channel simulation.

For any two quantum channels $\mathcal{N}$ and $\mathcal{M}$, the minimum error of simulation from $\mathcal{N}$ to $\mathcal{M}$ under Ω-assisted codes is defined as

$$\omega_\Omega(\mathcal{N},\mathcal{M}):=\frac{1}{2}\inf_{\Pi\in\Omega}\|\Pi\circ\mathcal{N}-\mathcal{M}\|_\diamond, \tag{1}$$

where $\|\cdot\|_\diamond$ denotes the diamond norm (for the definition see, e.g., [37]). The channel simulation rate from $\mathcal{N}$ to $\mathcal{M}$ under Ω-assisted codes is defined as

$$S_\Omega(\mathcal{N},\mathcal{M}):=\lim_{\varepsilon\to0}\inf\left\{\frac{n}{m}:\omega_\Omega\left(\mathcal{N}^{\otimes n},\mathcal{M}^{\otimes m}\right)\le\varepsilon\right\}, \tag{2}$$

where the infimum is taken over ratios $\frac{n}{m}$ with $n,m\in\mathbb{N}$. In this framework of channel simulation, the classical capacity $C(\mathcal{N})$ and the quantum capacity $Q(\mathcal{N})$ are given by

$$C(\mathcal{N})=S\left(\mathcal{N},\widehat{\mathrm{id}}_2\right)^{-1} \text{ and } Q(\mathcal{N})=S(\mathcal{N},\mathrm{id}_2)^{-1}, \tag{3}$$

where $\widehat{\mathrm{id}}_2$ is the 1-bit noiseless channel, id_2 is the 1-qubit noiseless channel and we eliminate subscript Ω for the case of unassisted codes.

If we consider simulating the given channel $\mathcal{N}$ via a m-dimensional noiseless quantum channel id_m, then the one-shot ε-error quantum simulation cost under Ω-assisted codes is defined as

$$S^{(1)}_{\Omega,\varepsilon}(\mathcal{N}):=\log\min\{m\in\mathbb{N}:\omega_\Omega(\mathrm{id}_m,\mathcal{N})\le\varepsilon\}. \tag{4}$$

The asymptotic quantum simulation cost is given by

$$S_\Omega(\mathcal{N})=\lim_{\varepsilon\to0}\lim_{n\to\infty}\frac{1}{n}S^{(1)}_{\Omega,\varepsilon}\left(\mathcal{N}^{\otimes n}\right). \tag{5}$$

III. Channel simulation via noisy channels

We show that the minimum error of simulation under NS-assisted and/or PPT-assisted codes can be given by SDPs. The one-shot ε-error quantum simulation cost under NS-assisted codes can also be given by an SDP.

Proposition 1 *For any two quantum channels $\mathcal{N}$ and $\mathcal{M}$ with corresponding Choi-Jamiołkowski matrices $J_\mathcal{N}$ and $J_\mathcal{M}$, the minimum error of simulation from $\mathcal{N}$ to $\mathcal{M}$ under NS-codes $\omega_{\mathrm{NS}}(\mathcal{N},\mathcal{M})$ is given by the following SDP,*

$$\min\ \|\mathrm{Tr}_{B_o}Y_{A_iB_o}\|_\infty \tag{6a}$$

$$\text{s.t. } Y_{A_iB_o}\ge J_{\widetilde{\mathcal{N}}}-J_\mathcal{M},\ Y_{A_iB_o}\ge0, \tag{6b}$$

$$J_{\widetilde{\mathcal{N}}}=\mathrm{Tr}_{A_oB_i}\left(J_\mathcal{N}^T\otimes\mathbb{1}_{A_iB_o}\right)J_\Pi, \tag{6c}$$

$$J_\Pi\ge0,\ \mathrm{Tr}_{A_oB_o}J_\Pi=\mathbb{1}_{A_iB_i},\qquad(\mathrm{CPTP}) \tag{6d}$$

$$\mathrm{Tr}_{A_o}J_\Pi=\frac{\mathbb{1}_{A_i}}{d_{A_i}}\otimes\mathrm{Tr}_{A_oA_i}J_\Pi,\qquad(\mathrm{A}\not\to\mathrm{B}) \tag{6e}$$

$$\mathrm{Tr}_{B_o}J_\Pi=\frac{\mathbb{1}_{B_i}}{d_{B_i}}\otimes\mathrm{Tr}_{B_iB_o}J_\Pi.\qquad(\mathrm{B}\not\to\mathrm{A}) \tag{6f}$$

To obtain $\omega_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N},\mathcal{M})$, we only need to add the PPT constraint $J_\Pi^{T_{B_iB_o}}\ge0$.

Proof: We denote the Choi-Jamiołkowski matrix of code Π as J_Π. Following the same idea in Refs. [9], [30], we know that the Choi-Jamiołkowski matrix of the effective channel $\Pi\circ\mathcal{N}$ is given by

$$J_{\widetilde{\mathcal{N}}}=\mathrm{Tr}_{A_oB_i}\left(J_\mathcal{N}^T\otimes\mathbb{1}_{A_iB_o}\right)J_\Pi. \tag{7}$$

For any two channels $\mathcal{N}_1, \mathcal{N}_2$ from A to B, the diamond norm of their difference, i.e., $\|\mathcal{N}_1 - \mathcal{N}_2\|_\diamond$ can be expressed as an SDP of the form [37]

$$\frac{1}{2}\|\mathcal{N}_1 - \mathcal{N}_2\|_\diamond = \min \|\operatorname{Tr}_B Y\|_\infty \tag{8a}$$

$$\text{s.t. } Y \geq J_{\mathcal{N}_1} - J_{\mathcal{N}_2},\ Y \geq 0, \tag{8b}$$

where $J_{\mathcal{N}_1}$ and $J_{\mathcal{N}_2}$ are the corresponding Choi-Jamiołkowski matrices. Combining with the constraints of code Π, we have the resulting SDP (6). The constraint in Eq. (6d) represent the CP and TP conditions of the operation Π. The constraints in Eqs. (6e) and (6f) represent non-signalling condition that A cannot signal to B and B cannot signal to A, respectively. ■

Corollary 2 *The minimum error to simulate a quantum channel $\mathcal{N}$ from noiseless quantum channel id_m under NS-codes $\omega_{\mathrm{NS}}(\mathrm{id}_m, \mathcal{N})$ is given by the following SDP,*

$$\min \|\operatorname{Tr}_{B_o} Y_{A_iB_o}\|_\infty \tag{9a}$$

$$\text{s.t. } Y_{A_iB_o} \geq J_{\widetilde{\mathcal{N}}} - J_{\mathcal{N}},\ Y_{A_iB_o} \geq 0, \tag{9b}$$

$$J_{\widetilde{\mathcal{N}}} \geq 0,\ \operatorname{Tr}_{B_o} J_{\widetilde{\mathcal{N}}} = \mathbb{1}_{A_i}, \tag{9c}$$

$$J_{\widetilde{\mathcal{N}}} \leq \mathbb{1}_{A_i} \otimes V_{B_o},\ \operatorname{Tr} V_{B_o} = m^2. \tag{9d}$$

To obtain $\omega_{\mathrm{NS}\cap\mathrm{PPT}}(\mathrm{id}_m, \mathcal{N})$, we only need to add the PPT constraint $-\mathbb{1}_{A_i} \otimes V_{B_o}^T \leq m J_{\widetilde{\mathcal{N}}}^{T_B} \leq \mathbb{1}_{A_i} \otimes V_{B_o}^T$.

Proof: Denote $J_m = \sum_{i,j=0}^{m-1} |ii\rangle\langle jj|_{A_oB_i}$ as the Choi-Jamiołkowski matrix of the operation id_m. The main idea is to exploit the symmetry of J_m and simplify the SDP (6).

Since J_m is invariant under any local unitary $U_{A_o} \otimes \overline{U}_{B_i}$, we can verify that $U_{A_o} \otimes \overline{U}_{B_i} J_\Pi U_{A_o}^\dagger \otimes \overline{U}_{B_i}^\dagger$ is also optimal if J_Π is optimal for SDP (6). Any convex combination of optimal solutions remains optimal. Thus, without lose of generality we can take

$$J_\Pi = \int dU \left(U_{A_o} \otimes \overline{U}_{B_i}\right) J_\Pi \left(U_{A_o} \otimes \overline{U}_{B_i}\right)^\dagger \tag{10}$$

$$= \frac{J_m}{m} \otimes C_{A_iB_o} + \left(\mathbb{1} - \frac{J_m}{m}\right) \otimes D_{A_iB_o}, \tag{11}$$

where the integral is taken over the Haar measure and C, D are operators on system A_iB_o.

Then, the condition in Eq. (6c) is equivalent to $J_{\widetilde{\mathcal{N}}} = mC$. The conditions in Eq. (6d) is equivalent to $C \geq 0$, $D \geq 0$ and $\operatorname{Tr}_{B_o}\left(C + \left(m^2-1\right)D\right) = m\mathbb{1}_{A_i}$. Since $J_{\widetilde{\mathcal{N}}}$ is the Choi-Jamiołkowski matrix of the effective channel, we have $\operatorname{Tr}_{B_o} J_{\widetilde{\mathcal{N}}} = \operatorname{Tr}_{B_o} mC = \mathbb{1}_{A_i}$ and $\operatorname{Tr}_{B_o} mD = \mathbb{1}_{A_i}$. This implies the condition in Eq. (6f) is trivial. The condition in Eq. (6e) is equivalent to $C + \left(m^2-1\right)D = \frac{\mathbb{1}_{A_i}}{d_{A_i}} \otimes \operatorname{Tr}_{A_i}\left(C + \left(m^2-1\right)D\right)$. Denote $V_{B_o} = \frac{m}{d_{A_i}} \operatorname{Tr}_{A_i}\left(C + \left(m^2-1\right)D\right)$. We have $J_{\widetilde{\mathcal{N}}} + \left(m^2-1\right)mD = \mathbb{1}_{A_i} \otimes V_{B_o}$. Eliminating variable D, we will have the desired SDP (9). ■

Proposition 3 *For any given channel $\mathcal{N}_{A_i \to B_o}$ and error tolerance $\varepsilon \geq 0$, the one-shot ε-error quantum simulation cost under NS-assisted codes is given by the following SDP,*

$$S_{NS,\varepsilon}^{(1)}(\mathcal{N}) = \frac{1}{2}\log \min \operatorname{Tr} V_{B_o} \tag{12a}$$

$$\text{s.t. } Y_{A_iB_o} \geq J_{\widetilde{\mathcal{N}}} - J_{\mathcal{N}},\ Y_{A_iB_o} \geq 0, \tag{12b}$$

$$J_{\widetilde{\mathcal{N}}} \geq 0,\ \operatorname{Tr}_{B_o} J_{\widetilde{\mathcal{N}}} = \mathbb{1}_{A_i}, \tag{12c}$$

$$J_{\widetilde{\mathcal{N}}} \leq \mathbb{1}_{A_i} \otimes V_{B_o}, \tag{12d}$$

$$\operatorname{Tr}_{B_o} Y_{A_iB_o} \leq \varepsilon \mathbb{1}_{A_i}. \tag{12e}$$

Proof: Note that $\|\operatorname{Tr}_{B_o} Y_{A_iB_o}\|_\infty \leq \varepsilon$ holds if and only if $\operatorname{Tr}_{B_o} Y_{A_iB_o} \leq \varepsilon\mathbb{1}_{A_i}$. From the definition of $S_{NS,\varepsilon}^{(1)}(\mathcal{N})$ in Eq. (4) and SDP (9), we have the desired result. ■

To match the exact definition of the one-shot quantum simulation cost, we need to apply a ceiling function to the optimal value $\sqrt{\operatorname{Tr} V_{B_o}}$ based on SDP (12). For simplicity, we ignore this extra step in the following discussion.

It is also worth mentioning that the one-shot quantum simulation cost under NS∩PPT-assisted codes is not an SDP, since the objective function m appears in the conditions $\operatorname{Tr} V_{B_o} = m^2$ and $-\mathbb{1}_{A_i} \otimes V_{B_o}^T \leq m J_{\widetilde{\mathcal{N}}}^{T_B} \leq \mathbb{1}_{A_i} \otimes V_{B_o}^T$ with different powers.

For any classical channel $\mathcal{N}(y|x)$, we can further simplify its quantum simulation cost to a linear program as follows,

$$S_{NS,\varepsilon}^{(1)}(\mathcal{N}) = \frac{1}{2}\log \min \sum V_y \tag{13a}$$

$$\text{s.t. } Y_{xy} \geq \widetilde{\mathcal{N}}(y|x) - \mathcal{N}(y|x), Y_{xy} \geq 0, \forall x, y, \tag{13b}$$

$$\widetilde{\mathcal{N}}(y|x) \geq 0, \forall x, y,\ \sum\nolimits_y \widetilde{\mathcal{N}}(y|x) = 1, \forall x, \tag{13c}$$

$$\widetilde{\mathcal{N}}(y|x) \leq V_y, \forall x, y,\ \sum\nolimits_y Y_{xy} \leq \varepsilon, \forall x. \tag{13d}$$

It is also worth mentioning that the zero-error quantum simulation cost was studied by Duan and Winter in Ref. [9]. We can recover their result by setting $\varepsilon = 0$, i.e.,

$$S_{NS,0}(\mathcal{N}) := \lim_{n\to\infty} \frac{1}{n} S_{NS,0}^{(1)}\left(\mathcal{N}^{\otimes n}\right) \tag{14}$$

$$= S_{NS,0}^{(1)}(\mathcal{N}) \tag{15}$$

$$= \frac{1}{2}\log \min\left\{\operatorname{Tr} V_{B_o} : J_{\mathcal{N}} \leq \mathbb{1}_{A_i} \otimes V_{B_o}\right\}. \tag{16}$$

The second line follows from the additivity of $S_{NS,0}^{(1)}(\mathcal{N})$, which can be verified by the SDP duality. The last line is known as the conditional min-entropy of $J_{\mathcal{N}}$.

IV. The Channel's Max-Information

Here we introduce the smooth max-information of a quantum channel and show that it has an operational meaning regarding the channel simulation cost. Furthermore, we obtain the asymptotic equipartition property (AEP) of the channel's smooth max-information from the well-known QRST.

The max-relative entropy [31] of $\rho \in \mathcal{S}_\leq(\mathcal{H}_A)$ with respect to $\sigma \geq 0$ is defined as

$$D_{\max}(\rho\|\sigma) := \log \inf\{t > 0 : \rho \leq t \cdot \sigma\}, \tag{17}$$

where $S_{\leq}(\mathcal{H}_A) := \{\rho \geq 0 : \operatorname{Tr}\rho \leq 1\}$ denotes the set of sub-normalized quantum states. The max-information that B has about A for $\rho_{AB} \in \mathcal{S}_{\leq}(\mathcal{H}_{AB})$ is defined as

$$I_{\max}(A:B)_\rho := \inf_{\sigma_B \in \mathcal{S}_=(\mathcal{H}_B)} D_{\max}(\rho_{AB} \| \rho_A \otimes \sigma_B), \quad (18)$$

where $S_=(\mathcal{H}_B) := \{\rho \geq 0 : \operatorname{Tr}\rho = 1\}$ denotes the set of quantum states. For any quantum channel $\mathcal{N}_{A' \to B}$ we define the max-information of $\mathcal{N}$ as

$$I_{\max}(A:B)_{\mathcal{N}} := I_{\max}(A:B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})}, \quad (19)$$

where $\phi_{AA'}$ is a purification of a full rank state $\rho_A \in S_=(\mathcal{H}_A)$. As the following argument shows this definition does actually not depend on the input state ρ_A. From the definitions, we have

$$I_{\max}(A:B)_{\mathcal{N}} = \log \inf t \quad (20a)$$
$$\text{s.t. } \mathcal{N}_{A'\to B}(\phi_{AA'}) \leq t\rho_A \otimes \sigma_B, \quad (20b)$$
$$\sigma_B \in \mathcal{S}_=(\mathcal{H}_B). \quad (20c)$$

Denote $J_{\mathcal{N}}$ as the Choi-Jamiołkowski matrix of the channel $\mathcal{N}$. Then, it holds that

$$\mathcal{N}_{A'\to B}(\phi_{AA'}) = \sqrt{\rho_A} J_{\mathcal{N}} \sqrt{\rho_A}, \quad (21)$$

which implies that the condition $\mathcal{N}_{A'\to B}(\phi_{AA'}) \leq t \cdot \rho_A \otimes \sigma_B$ is equivalent to $J_{\mathcal{N}} \leq t \cdot \mathbb{1}_A \otimes \sigma_B$. Thus, the definition in Eq. (19) is independent of the state ρ_A.

With Eq. (16) we can write the one-shot zero-error simulation cost as the channel's max-information:

$$S^{(1)}_{NS,0}(\mathcal{N}) = \frac{1}{2} I_{\max}(A:B)_{\mathcal{N}}. \quad (22)$$

In the following, we show this relation beyond the zero-error case. Define the smooth max-information of a given channel $\mathcal{N}_{A'\to B}$ as

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}} := \inf_{\substack{\frac{1}{2}\|\widetilde{\mathcal{N}}-\mathcal{N}\|_\diamond \leq \varepsilon \\ \widetilde{\mathcal{N}} \in \text{CPTP}(A':B)}} I_{\max}(A:B)_{\widetilde{\mathcal{N}}}, \quad (23)$$

where $\text{CPTP}(A':B)$ denotes the set of all the CPTP maps from A' to B. We show that the one-shot channel simulation cost is given by the channel's smooth max-information. This provides the operational meaning of this new max-information.

Theorem 4 *For any quantum channel $\mathcal{N}_{A'\to B}$ and given error tolerance $\varepsilon \geq 0$, we have*

$$S^{(1)}_{NS,\varepsilon}(\mathcal{N}) = \frac{1}{2} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}}. \quad (24)$$

Proof: Note that the constraints $J_{\widetilde{\mathcal{N}}} \geq 0$, $\operatorname{Tr}_B J_{\widetilde{\mathcal{N}}} = \mathbb{1}_A$ in Eq. (12c) uniquely define a CPTP map $\widetilde{\mathcal{N}}$ according to Choi-Jamiołkowski isomorphism. Applying the SDP (8) for diamond norm, we have

$$S^{(1)}_{NS,\varepsilon}(\mathcal{N}) = \frac{1}{2} \log \min \operatorname{Tr} V_B \quad (25a)$$
$$\text{s.t. } J_{\widetilde{\mathcal{N}}} \leq \mathbb{1}_A \otimes V_B, \quad (25b)$$
$$\frac{1}{2}\|\widetilde{\mathcal{N}} - \mathcal{N}\|_\diamond \leq \varepsilon, \quad (25c)$$
$$\widetilde{\mathcal{N}} \in \text{CPTP}(A':B). \quad (25d)$$

From Eqs. (14) and (22), we know that

$$I_{\max}(A:B)_{\mathcal{N}} = \log \min \operatorname{Tr} V_B \quad (26a)$$
$$\text{s.t. } J_{\mathcal{N}} \leq \mathbb{1}_A \otimes V_B. \quad (26b)$$

Combining SDPs (25) and (26), we find the desired result. ■

Corollary 5 *The channel's smooth max-information has the asymptotic equipartition property, i.e., for any $\varepsilon \in (0,1)$*

$$\lim_{n\to\infty} \frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = I(A:B)_{\mathcal{N}}, \quad (27)$$

where $I(A:B)_{\mathcal{N}} = \max_{\rho_A} I(A:B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})}$ is the mutual information of the channel $\mathcal{N}$, and $\phi_{AA'}$ is a purification of $\rho_A \in S_=(\mathcal{H}_A)$.

To see this, let us recall the quantum channel capacity is given by the optimal rate of using $\mathcal{N}$ to simulate the qubit identity channel id_2, while the channel simulation cost is given by the optimal rate of using id_2 to simulate the channel $\mathcal{N}$. Thus, it is clear that

$$Q_E(\mathcal{N}) \leq Q_{NS}(\mathcal{N}) \leq S_{NS}(\mathcal{N}) \leq S_E(\mathcal{N}), \quad (28)$$

where the above notations represent entanglement-assisted quantum capacity (Q_E), NS-assisted quantum capacity (Q_{NS}), NS-assisted simulation cost (S_{NS}) and entanglement-assisted simulation cost (S_E), respectively. According to the QRST we have $Q_E(\mathcal{N}) = S_E(\mathcal{N})$ and hence Eq. (28) collapses to equalities. Thus, we have

$$S_{NS}(\mathcal{N}) = Q_E(\mathcal{N}) = \frac{1}{2} I(A:B)_{\mathcal{N}}, \quad (29)$$

where the second equality was proved in Ref. [3]. Hence,

$$\lim_{n\to\infty} \frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = \lim_{n\to\infty} \frac{2}{n} S^{(1)}_{NS,\varepsilon}\left(\mathcal{N}^{\otimes n}\right) \quad (30)$$
$$= 2 \cdot S_{NS}(\mathcal{N}) \quad (31)$$
$$= I(A:B)_{\mathcal{N}}, \quad (32)$$

where the first line follows from Eq. (24). The second line follows since the simulation cost is a strong converse rate [4].

Proposition 6 *The channel's smooth max-information is monotone under composition with CPTP maps, i.e., for any CPTP maps $\mathcal{N}_{A_1'\to B_1}$, $\mathcal{F}_{A_0'\to A_1'}$ and $\mathcal{T}_{B_1\to B_0}$,*

$$I^{\varepsilon}_{\max}(A_0:B_0)_{\mathcal{T}\circ\mathcal{N}\circ\mathcal{F}} \leq I^{\varepsilon}_{\max}(A_1:B_1)_{\mathcal{N}}. \quad (33)$$

Proof: From Thm. 4, we only need to show that

$$S^{(1)}_{NS,\varepsilon}(\mathcal{T}\circ\mathcal{N}\circ\mathcal{F}) \leq S^{(1)}_{NS,\varepsilon}(\mathcal{N}). \quad (34)$$

Suppose the optimal NS-assisted code for the channel $\mathcal{N}$ is $\Pi_{A_1'B_2\to A_2B_1}$ and $S^{(1)}_{NS,\varepsilon}(\mathcal{N}) = \log m$, $\frac{1}{2}\|\Pi\circ\text{id}_m - \mathcal{N}\|_\diamond \leq \varepsilon$. Then, $(\mathcal{F}\otimes\mathcal{T})\circ\Pi$ is also an NS-assisted code and

$$\frac{1}{2}\|\left((\mathcal{F}\otimes\mathcal{T})\circ\Pi\right)\circ\text{id}_m - \mathcal{T}\circ\mathcal{N}\circ\mathcal{F}\|_\diamond$$
$$\leq \frac{1}{2}\|\Pi\circ\text{id}_m - \mathcal{N}\|_\diamond \leq \varepsilon. \quad (35)$$

Thus, we have $S^{(1)}_{NS,\varepsilon}(\mathcal{T}\circ\mathcal{N}\circ\mathcal{F}) \leq \log m = S^{(1)}_{NS,\varepsilon}(\mathcal{N})$. ■

This result is compatible with the intuition that we need less resources to simulate a quantum channel with higher noise.

V. Discussion

We have used the QRST to obtain Eq. (27). However, from the dual perspective, a direct proof of Eq. (27) without relying on the QRST would also imply the equation $Q_{NS}(\mathcal{N}) = S_{NS}(\mathcal{N})$. This corresponds to a (slightly weaker) version of the QRST and thus directly proving Eq. (27) may shine some light on a simple proof of the QRST itself. For this purpose we need to better understand the relation between the channel's smooth max-information and other measures of max-information defined for quantum states [5], [28]. Finally, since the entanglement-assisted capacity allows a single-letter characterization it is natural to consider a second-order refinement thereof. A second-order expansion of an achievable rate was established in [38] but no matching second-order converse bound is known. Our one-shot NS-assisted quantum channel simulation cost may provide some insights in this direction.

Note that the channel simulation cost operationally provides converse for the channel capacity. However this approach does not provide tighter bound than the NS-assisted capacity in the one-shot and asymptotic setting (see, e.g., [19], [26], [30]).

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